Notes for the 2024 Young Topologist Meeting

Dév Vorburger

August 23, 2024

Contents

1 Introduction

These are the notes I took during the 2024 YTM in Münster, I want to express my deepest gratitude to all the organizers, speakers and in general all attendees for a very instructive, enriching and interesting week.

These notes surely contain many mistakes, omissions and spelling mistakes, the reader is invited to assume that all of these are my own, and not mistakes of the speakers. Regarding spelling, I in particular want to specify that misspelled or uncapitalized names are not meant to be taken as a sign of disrespect, but rather as an indication of difference between the speaker's talking speed and my writing speed.

I did my best to specify lines where I most definitely missed something using **bold text**. My final apology is for a mix of languages, though 99% of what I wrote is English, so I am ending this introduction on a rather small apology.

2 Ishan Levy's mini course on Telescopic computations in Algebraic K-theory

2.1 Talk 1

Things related to the disproof of the telescope conjecture. Computational point of view.

Some references:

1. On topological cyclic homology Nikolas and Scholze.
2. K-theoretic counter examples to the Ravenel's telescope conjecture.

The Telescope conjecture belongs to stable homotopy theory, which studies e.g $\pi_i^s(\mathbb{S})$. We know some basic computations$\pi_k(\mathbb{S}) =$

1. 0 for $k < 0$
2. $\mathbb{Z}$ for $k = 0$
3. A finite abelian group for $k > 0$

We often localize at a prime p for further studies, we work in the category of spectra Sp.

There is a notion of spectrum of spectra $\mathrm{Spec}^{\Delta}(\mathbb{S}_p)$ where p is some prime p (perfectly analogous, but different to $\mathrm{Spec}(\mathbb{Z}_{(p)})$).

Definition 1. *A ring spectrum R is a field if $\pi_*(R)$ is a gradded field.*

We say that R and R' have the same characteristic if $R \otimes R' \neq 0$ (by tensor product we mean smash product).

One question is what characteristics of fields can we find. And it turns out: We can realize them all in $Sp_{(p)}$ by using the Morava K-theories $K(n)$. These are in general quite hard to understand, but we have a relatively nice description 0, 1, ∞ and of $\pi_*(K(n))$ using Witt vectors **I think**. Each of these $K(n)$ corresponds to the "residue field" at an element $p_n \in \mathrm{Spec}^{\Delta}(\mathbb{S}_{(p)})$.

One trick for constructing elements of the stable homotopy groups is using a map from a finite spectra $v : V \to \mathbb{S}$, then use the following diagram

$$\Sigma^n \mathbb{S} \to \Sigma^m V \to V \to \mathbb{S}$$

(where n and m aren't arbitrary but apparently not important right now, the first map I think comes from initiality of $\mathbb{S}$ and the second I believe is some natural "collapse" map?) From this we will take a lot of powers and if we are careful we might get a ton of elements of the stable homotopy groups.

So question: What (V, v) do we want to consider. First of all v shouldn't be nilpotent and second of all we should consider central v (whatever that means).

We can construct an example by taking the cofiber of $\mathbb{S} \xrightarrow{p} \mathbb{S}$, which is $\mathbb{S}/p$. This map has a self map of degree $2p - 1$ (i.e a map $\Sigma^{2p-1}\mathbb{S}/p \to \mathbb{S}/p$) call this map v_1, take the cofiber, we get a space $\mathbb{S}/p, v_1$. We then do this repeatedly and get maps v_n, a map of degree $2p^n - 1$ from the cofiber of v_{n-1} to itself.

There is a theorem of Hopkins and Smith which explains the point of these v_n, in particular we use these to construct a spectra $T(n)$ which is the telescope **Unclear whether my terminology is correct**. Now we have two nice families of spectra $T(n), K(n)$ the telescopes and the spectra of Morava K-theories. The former is great cause $L_{T(n)}Sp$ contains info about stable homotopy groups

up to v_n-periodicity. And $K(n)$ is great because stuff in $L_{K(n)}Sp$ is very computable using ideas/tools from arithmetic geometry. That these two bousfield localizations actually yield the same category (this Ravenel's telescope conjecture). The raison d'être of the conjecture (**I think**) is that in the case $n = 1$ this is true, but for $n \geq 2$ this is false.
The difference is proven to exist by using "trace methods with cyclic homology" ... whatever that means. What they show is that $L_{T(n)}K(BP < n-1 >^{h\mathbb{Z}})$ is not $K(n)$ local (which shows that the two categories are different, as this is obviously $T(n)$ local).
By initiality there is a natural map $L_{T(n)}\mathbb{S} \to L_{T(n)}K(BP < n-1 >^{h\mathbb{Z}})$ which we can use... to do stuff?

There is a theorem by a bunch of people which says

$$\frac{\Sigma_1^n \operatorname{rk_p}(\pi_i(\mathbb{S}))}{n} \geq O(\log(n))$$

Now the thing which helps, is that in the case of $BP < n-1 >^{h\mathbb{Z}}$ the algebraic K-theory $K(-)$ (so this spectrum is a ring spectra) is the same as topological cyclic homology $TC(-)$.

What we need to show is that some map involving the spectra $T(n) \otimes TC(BP < n-1 >^{h\mathbb{Z}})$ or spaces that look like it, which are weak equivalences but not $K(n)$-locally.

Now let's start discussing these functors $K, TC : \text{RingSpectrum} \to Sp$. There is a natural transformation comparing them $K \to TC$ called cyclotomic trace.

There is a neat theorem which says that $L_{T(n)}K(R)$ depends only on $L_{T(n)\otimes T(n-1)}R$. This is nice because usually $K(R)$ is really annyoing to compute, but maybe not in this case?

There is a theorem (Dundas G... Mcarthy) which says that the natural maps $TC(R) \leftarrow K(R) \to K(\pi_*(R))$ and $TC(R) \to TC(\pi_*(R)) \leftarrow K(\pi_*(R))$ assemble into a pullback square. The speaker uograded this theorem to the case where the ring has a $\mathbb{Z}$ action. In which case this also a pullback square after taking homotopy fixed points. But we may lose some connectivity.

So TC is a ring spectrum invariant, which is constructed from THH which is another ring spectrum (rings are assumed to be $\mathbb{E}_1$). View R as an R bimodule, which is the same as an $R \otimes R^{op}$ module. Then we define

$$THH(R) = R \otimes_{R\otimes R^{op}} R$$

We can also replace the second R by an R bimodule M to get a THH with coefficients M.
In the commutative case we also have (in the category of commutative ring spectra)

$$THH(R) = \varinjlim_{S^1} R$$

This can be generalized to general symmetric monoidal categories, so we get a functor $THH_C : Alg(C) \to C$ for C symmetric monoidal.

For some reason, THH_C has an S^1 action (I don't know for what C, any symmetric monoidal category???)
In spectra $THH(R)$ has extra structure, in particular a map, called the Frrobenius maps $\phi_p : THH(R) \to THH(R)^{tC_p}$, one for each prime p, this map is S^1-equivariant and the superscript tC_p means the "tate construction".

Spectra with an S^1 action and a collection of maps for each prime $X \to X^{tC_p}$ which are S^1 equivariant assemble into a category, the category of cyclotomic spectra (unclear whether my definition is correct). So THH is a functor RingSp $\to$ CycSp. We use this category to define topological cyclic homology:

$$TC(X) = \operatorname{Mor}_{\text{CycSp}}(THH(\mathbb{S}, X))$$

And so for a ring R we denote by $TC(R)$ the spectrum $TC(THH(R))$.

For example, there is a fundamental calculation by Béckstedt who computed $TC(\mathbb{F}_p)$. First we need to understand $THH(\mathbb{F}_p) = \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathbb{F}_p^{op}} \mathbb{F}_p$. We can (relatively easily) understand the homotopy of $THH(\mathbb{F}_p)$ to be the Dual Steenrod algebra mod p, which for $p = 2$ is known to be a polynomial algebra $\mathbb{F}_p[\tau_i]$ with each τ_i a generator in degree $2p^i - 1$. In general though it is given by $\mathbb{F}_p[\sigma^2 p]$. From this we can deduce that as $\mathbb{F}_p - \mathbb{E}_2$ algebras $THH(\mathbb{F}_p) \cong Free_{\mathbb{E}_2 - \mathbb{F}_p}(\sigma^2 p)$.

2.2 Talk 2

The algberaic K-theory of KU.
Reference:

1. LL topological Horschild homology of the image of J.
2. HW redshift and multiplication for truncated brown peterson spectra.
3. HRW a motivic filtration on the topological cyclic homotopy of comm ring. spectra.

Well actually we won't study algerbraic K-theory, but TC, and we also lied about studying KU, but instead we will study a connective cover of a summand, called the adam's summand, called $l = BP < 1 >$. It's homotopy ring is $\mathbb{Z}_{(p)}[v_1]$ (this is related in some sense to homotopy ring of KU, but didn't quite catch how).

Recall TC is built from THH, so a good first computation is $THH(l)$, and we can mod out (by taking an appropriate cofiber) by some of the stuff we already know, i.e the prime p and the element v_1 of the homotopy ring. It turns out this will be the same as $THH(l, l/(v_1, p))$ which is the same as $THH(l, \mathbb{F}_p)$. Now l has an Adams filtration, which gives a grading on l, and we can take the associated graded ring $gr(l) \cong \mathbb{F}_p[v_0, v_1]$, where all these elements have some grading.

Whenever we have a filtered spectrum, we have a spectral sequence, the E_1 page is the associated graded **smth?**, so the E_1 page is $THH(\mathbb{F}_p[v_0, v_1], \mathbb{F}_p)$ and it converges to $THH(l, \mathbb{F}_p)$. One property of THH is that it is a symmetric monoidal functor, i.e preserves tensor products of algebras. We can use this by noticing $\mathbb{F}_p \otimes \mathbb{S}[v_0, v_1]/v_0 v_1$. Now $THH(\mathbb{F}_p) = \mathbb{F}_p[\sigma^2 p]$ is a "well known" computation, and so all we need to do is understand $THH(\mathbb{S}[v_0, v_1]/v_0 v_1)$, and for some reason which presumably is clear if you know more abt THH and stuff, is that this is the same as $HH_{\mathbb{F}_p}(\mathbb{F}_p[v_0, v_1])$ and this is a classical invariant (i.e invariant of rings, no need for spectra or any higher stuff). This is relatively computable, see:

Theorem 1. *(HKR) We have*

$$\pi_*(HH(k[x])) = k[x] < dx > .$$

So applying this theorem to our case yields $\mathbb{F}_p[\sigma^2 p] < dv_0, dv_p >$ (recall we are doing this to understand a certain spectral sequence).

There is some magical trickery which allows one to define a map $\Sigma X \to \sigma e \otimes X \to THH(X)$ in any symmetric monoidal category. Doing this for $\mathbb{E}_\infty$ rings, yields a map $\Sigma R \to S^1 \otimes R \to THH(R)$. Let's call this map d, it turns out this map is natural, let's study it for the initial $\mathbb{E}_\infty$ ring, which is the sphere spectrum. In that case d will have to be 0, by initality and naturality, this means that other d, factor through the cofiber of the suspension of the initial map $\mathbb{S} \to R$. We call the map out of the cofiber to be σ^2. And then a lot of stuff was said whose point wholly elluded me.
A bunch of stuff are modules over other stuff, create a nice diagram relating all the stuff and then get some relations between different algebras and then for some reason we are happier than we were before.
For example we have a fundamental computation in the homotopy ring of $THH(\mathbb{F}_p)$ which is $(\sigma^2 p)^{p^i} = \sigma^2 v_i$, and this apparently has many consequences when computing TC. Maybe the raison d'être of all

this stuff is to understand the Tor spectral sequence of $THH(\mathbb{F}_p) = \mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p$, which is what is used to compute the topological Horschild homology of $\mathbb{F}_p$.

All of the above reasoning (which I didn't really write out) is sufficiently general to be done in filtered spectra, thus giving some spectral sequence information, which is nice cause that is what we are computing. I guess the moral is that "geometric intution" can be transfered $Top \to Sp \to FilteredSp$ and in the latter category this has powerful computational ramifications, which in turn is cool because computations have implications. In our case the whole stuff I didn't understand was to compute a THH, this being done, we can try to understand TC, which requires understanding which is S^1 equivariant.

One cool thing which you do, for some reason, is to take the one point compactification of the standard representation of S^1, which is $S^1 \to U^1$. The point of this is beyond me, but it is plenty cool. I think it is because in an S^1 equivariant setting, this is more natural than S^1 for purposes such as smashing. Maybe not quite more natural, but containing ever so slightly more information.

There is a Bockstedt spectral sequence which has implications for homotopy fixed point. Which is nice because topological cyclic homology has a lot to do with homotopy fixed points and THH has a circle action. The connection is given bc TC is the equalizer of maps out of $THH(-)^{hS^1}$ into $THH(-)^{tS^1}$. One map is the cannonical one, and the other one is the Frobenius map And to understand this you take, among other things, homotopy fixed point spectral sequences.

2.3 Talk 3

The height 2 telescope conjecture.

Recall: where were we last time: how to compute the homotopy groups of THH of the adams summand $BP < 1 >$ quotiented out by some stuff we understand, and this is with the goal of understanding the topological cyclic homology of the same space, quietened out by the same "trivial" stuff. Taking homotopy fixed points before taking TC yields a telescope spectrum $TC(l^{h\mathbb{Z}})/(p, v_1)[v_2^{-1}]$ which is the contradiction to the height 2 telescope conjecture due to a comparaison map with $TC(l)^{h\mathbb{Z}}/(p, v_1)[v_2^{-1}]$ which is $K(2)$-local, so if the comparison map is not an equivalence, we are good.

The computation of that map is similar to the computation $THH(l)/(p, v_1) \to TC(l)/(p, v_1)[v_2^{-1}]$, and this has been done in the 2000s by Ausoni and Rognes, thus in priniciple could have been done all those years ago. But a more recent method by **smbdy** turned out to be used by Ishan Levi and co.

Now to compute TC given THH, there is an equalizer diagram relating them, by taking an equalizer of the cyclotomic frobenius map and the cannonical map from the S^1-homotopy fixed point to the S^1 tate construction. One tool for this computation:

Theorem 2. *(THH Segal conjecture) There is a map $THH(l)/(p, v_1) \to THH(l)/(p, v_1)^{tC_p}$ which is equivalent to a map $THH(l)(p, v_1)[\sigma^2 v_2^{-1}]$. Which implies that the Frobenius map is an isomorphism in large degrees.*

So we want to understand $THH(l)^{tC_p}/(p, v_1)$, we can make this easier by quotienting by v_2 as well. We know its homotopy groups because I think we already did it. And so we can understand the E_2 page of the Tate spectral sequence, because we know to what it converges. To this we only need to understand the class σ^2 equivariantly and the THH Segal conjecture. With some trickery this reduces to understanding a certain d_2 differential. We can only understood things up to units, which normally isn't important, but it is important in some specific construction.

We first discuss stuff C_p equivariantly, but the goal is S^1-equivariantly. But it turns out in this case C_p-equivariance determines S^1 equivariance, as we have maps $X^{tS^1} \to X^{tC_p}$. And so by a commutative diagram it suffices to understand the map I just mentioned to understand a Frobenius map.

Then he uses something called "cannonical vanishing", which I think says something about a frobenius map. And then some reasoning about connectivity of the involved spaces. Following this a bunch of reasoning is done which I do not understand and so decided not to copy. I will just list some things he mentions.

There is a v_2 Bockstein spectral sequence. Then some trickery to compare THH with some other THH and one related to HH.

3 Arunima Ray's mini course

3.1 Talk 1

Three lectures:

1. Intro to 4-Manifolds and slice knots/links
2. Applications to smooth structures
3. Connection to surgery theory

This is a talk in low dimensional topology which is a subset of geometric topology. The lectures will be in increasing order of difficulty.

Recall: closed means compact and without boundary.

Why are we interested in 4-manifolds? Assuming manifolds are interesting, 4-dimensional manifolds are particular among these. In particular, the behaviour varies quite a bit depending on dimension

1. Dimension $n = 1, 2$ and 3 (See Rado 1925, Moise 1952)
   (a) The fundamental group is quite restricted
   (b) Every topological manifold has a unique smooth structure.
   (c) $(-)$
   (d) $\mathbb{R}^n$ has a unique smooth structure.
   (e) The Poincaré conjecture is true. (a homotopy sphere is a smooth sphere)
2. Dimension $n = 4$
   (a) Every finitely presented groups is the fundamental group of some closed manifold of dimension n
   (b) There exists topological manifolds with no smooth structure at all, and there exists some with many smooth structures (see Siebenmann Rochlin in the seventies)
   (c) There exist closed topological manifolds admitting countably many smooth structures (and for closed manifolds there can't be more than countably many). (see Fintushel Stein 1998) one example is the $K3$ surface.
   (d) $\mathbb{R}^n$ has unaccountably many smooth structures. (Taubes 87)
   (e) The topological Poincaré conjecture is true (See Freedman 1982), but the differenital Poincaré conjecture is still open
3. Dimension $n = 5$ or higher
   (a) Every finitely presented groups is the fundamental group of some closed manifold of dimension n
   (b) There exists topological manifolds with no smooth structure at all, and there exists some with many smooth structures. (see Siebenmann Rochlin in the seventies)
   (c) Every closed topological manifold has finitely many smooth structures (possibly none) (See Kirby and SiebenMann in the seventies), this is via something called "product structure theorem" which translates the problem into homotopy theory
   (d) $\mathbb{R}^n$ has a unique smooth structure. (Stallings 1962)
   (e) The topological Poincaré conjecture is true (Smale 1960, Newmann 1966). The differenital Poincaré conjecture is generally false.

Another way to distinguish dimensions is what tools are available to us. For example in high dimension (higher than 5) we have both the topological and differential s-cobordism theorem (see Smale, Bouden, Mazur, Stallings, Kirby, Siebenmann) and the topological and differential surgery (exact) sequence (Browder, Novikov, Sullivan, Kirby, Siebenmann).

These two tools are really powerful "hammers", which make high dimensional topology in some sense "easy".

In dimension 4, we do not have the differential s-cobordism theorem or the differential surgery sequence (see Donaldson). But we do have the topological s-cobordism theorem and the topological surgery (exact) sequence for some fundamental groups (see Freedman 1982, Freedman-Quinm 1990).

As one can see, when we are not considering smooth structures, 4-manifolds are very algebraic/homotopic, when we consider smooth structures, the story changes quite drastically. And in addition we have access to very geometric theorems/techniques because of how low dimensional the situation is.

Okay now let's talk about knots/links.

Definition 2. *A knot is a smooth embedding $S^1 \to S^3$ and a link is a smooth embedding $\sqcup S^1 \to S^3$. (everything is oriented)*

Proposition 1. *A knot is trivial if and only if it bounds a smoothly embedded D^2.*

Definition 3. *A knot K is said to be smoothly slice if we can extend the map $\iota \circ K : S^1 \to S^3 \to D^4$ to a smooth embedding $D^2 \to D^4$.*

Intuitievely, slice knots generalize the unknot in so far as the disc which it bounds is allowed to "leak" into an extra dimension.
One example is given by the steveadore knot **no idea if I spelt it correctly**.

Definition 4. *A knot K is said to be topologically slice if we can extend the map $\iota \circ K : S^1 \to S^3 \to D^4$ to a locally flat embedding $D^2 \to D^4$.*

Both the notion of smooth and topological slice can easily be generalized to links. A slice link $L : \sqcup S^1 \to S^3$ is freely slice if $\pi_1(B^4 \setminus \sqcup D^4)$ is a free group.

Theorem 3. *(proven in next talk) Every knot which is topologically slice but not smoothly slice gives rise to an exotic smooth structure on $\mathbb{R}^4$.*

Theorem 4. *The topological $4D$ surgery sequence is exact for all fundamental groups if and only if every "good boundary link" is freely slice.*

3.2 Talk 2

The talk starts with discussing that the steveadore knot is smoothly slice (and not trivial), but because this is visual and I don't want to draw in tex.

To visualize 4 dimensional space, there will be some flip book stuff going on, say we have 4-coordinates (x, y, z, t) we visualize one t at a time. For embeded knots this works pretty well, because we can use planar diagrams to visualize (x, y, z, t_i) (fixed t) as $\mathbb{R}^2$.

Historically: Resolution of singularities. Let Σ be the solution set for $z^2 = w^3$ in $\mathbb{C}^2$. Take a small epsilon ball B^4_ϵ about the origin, now we can consider $\Sigma \cap \partial B^4_\epsilon$ is a 1-dimensional object in S^3, i.e a knot, it turns out it is in fact the trefoil knot. Cheating and drawing Σ in $\mathbb{R}^2$, we see there is a singularity at the origin. And we see that one way to resolve this singularity would be if $\Sigma \cap B^4_\epsilon$ were slice. This is a historical motivation.

Problems of this type are often interesting, sadly in this case it is not slice. How can we see this? With the Alexander polynomial.

Definition 5. *Let $K \subset S^3$ be a knot, by Alexander duality $H_1(S^3 \backslash K) \cong \mathbb{Z}$, so denote by $\widetilde{S^3 \backslash K}$ the cover of the knot complement corresponding to the subgroup $[\pi_1(X), \pi_1(X)]$ ($X = S^3 \backslash K$). This covering space has by deck transformation an action of $\mathbb{Z}$. By homology with coefficients the following is defined*

$$H_1(\tilde{X} m \mathbb{Z}[t, t^{-1}])$$

We call it the Alexander module of the knot. It turns out to be a torsion module over $\mathbb{Z}[t, t^{-1}]$. Then the Alexander polynomial is $\Delta_K(t) = ord(\text{Alexander-Module(K)})$. I think this can be interpreted as a generator for the ideal of $\mathbb{Z}[t, t^{-1}]$ annihilating the Alexander module.

We can replace $\mathbb{Z}$ by $\mathbb{Q}$ to get the rational Alexander module, yielding the same notion of Alexander polynomial (in which case the twisted coefficient ring is a PID, so ideals have a generator well defined up to multiplication by units, which can be nice). It is only defined up to multiplication by unit.

Why is this interesting? By a theorem of Fox-Milnor which says that for a topologically slice knot K the Alexander polynomial has the form $f(t)f(t^{-1})$. And by an "easy" computation, we have that the Alexander polynomial of the (right oriented) trefoil knot is $t - 1 + t^{-1}$ is not of the required form, thus not slice.
We also have a kind of inverse, which is a theorem of [Freedman-Quinn,90], which is that if the Alexander polynomial is 1, then the knot is Topologically slice.

A theorem of [Quinn, 86], locally flat submanifolds of 4-manifolds have normal vector bundles. This means something for locally flat embeddings $\Delta : D^2 \to B^4$, there exists a neighborhood of Δ homeomorphic to $\Delta \times D^2$.

Unsuprisingly there exists topologically embedded discs in B^4 which aren't locally flat ad some which are locally flat but not smooth.

One example which is interesting, but I didn't quite catch why: the Whitehead doubles of a knot, denoted by $Wh(K)$, it turns out these always have trivial Alexander polynomial, thus are always topologically slice.

It is now time for the promissed result:

Theorem 5 (Gompf)**.** *Every knot which is topologically slice but not smoothly slice gives rise to an exotic smooth structure on $\mathbb{R}^4$.*

One example is given by by the Whitehead double of the right handed trefoil knot.

Proof. Given a $K \subset S^3$ a topologically slice knot, then by the Trace embedding lemma, we get a a topological embedding $X_k \to \mathbb{R}^4$. One can relatively easily believe that X_k is smooth, so we can smooth $\phi(X_k)$.
We proceed by way of contradiction. Now we can consider $\mathbb{R}^4 \backslash \operatorname{Int}(\phi(X_k))$ is a connected, non compact 4-manifold with a smooth structure on the boundary (this isn't necessairly obvious). By a result of [Quinn, 80s] we can give the entire manifold a smooth structure. In total we get a smooth structure on $\mathbb{R}^4$, call $\mathbb{R}^4$ with this smooth structure $\mathcal{R}$. Suppose we have a dffeomoprhism $\mathcal{R} \cong \mathbb{R}^4$, we would have a smooth embedding $X_k \to \mathbb{R}^4$, which would imply that K is smoothly slice by the trace embedding lemma, which is the desired contradiction. ☐

key tool for the proof:

Lemma 1. *(Trace embedding Lemma) A knot $K \subset S^3$ is smoothly/topologically slice iff the trace X_K can be embedded in $\mathbb{R}^4$ smoothly/topologically.*

Proof. ($\Rightarrow$) This is done by a picture, which I will not be drawing.
($\Leftarrow$) It is fact that any two orientation preserving (smooth/topological) embedding of B^4 are ambient isotopic. So we use this result on the embedding, to get an embedding which fills the top hemisphere, and thus kinda gives a result (the picture helps, which I did not draw). □

Okay but what is a trace

Definition 6. *Given a knot $K \subset S^3$, the trace X_K is what we get by attaching a $\partial D^2 \times D^2$ to a tubular neighbourhood of K seen as a subset of B^4. In words we attach a two handle to a four-ball at the knot embedded in the surface of the four-ball.*

3.3 Talk 3

There exists an idea to use knots/links and their slice properties to disprove the smooth 4-dimensional Poincarré conjecture, essentially done by removing a 4-ball from a 4-dimensional manifold, putting a knot in the boundary S^3, then seeing if this knot is slice in this specific case. Then maybe we can find a knot which is slice in an exotic sphere but not in a true sphere, thus showing that the exotic 4-sphere is genuinely exotic.

Today we talk about the surgery conjecture and its connection to knots and links.
Recall the result of Freedman and Quinn which says that a knot of Alexander polynomial 1 is topologically slice. Which is the same as saying that the alexander module is trivial, which by some local coefficients and dimension Hurewicz stuff is equivalent to saying that $\pi_1(S^3\backslash K)^{(1)}/\pi_1(S^3\backslash K)^{(2)} = 0$, i.e the first derived subgroup $()^{(1)}$ is perfect.

Definition 7. *An n-complete link L is a good boundary link if there exists a map $\phi : \pi_1(S^3\backslash L) \twoheadrightarrow Free(n)$ whose kernel is perfect.*

For example every Alexander polynomial 1 knot is a good boundary link. Or If L has pairwise linking number 0, then the Whitehead double is a good boundary link.

The exactness of the surgery sequence being exact for all π_1 implies that every good boundary link is freely slice. This is not that surprising, what is surprising is that it goes the other way.

What is the surgery sequence though?

Let X be a topological closed 4-manifold, then its (4-dimensional) surgery sequence is

$$\mathcal{S}(X) \to \mathcal{N}(X) \to L_4(\pi_1(X))$$

The L_4 group is purely algebraic, which consists of non singular quadratic forms ($\mathbb{Z}[\pi_1(X)]$-module equipped with two forms) which satisfy some properties quotiented by stably hyperbolic quadratic forms. Usually $\mathcal{S}$ is the one we are interested in, it is the group of closed topological 4 manifolds over X up to cobordism. And $\mathcal{N}$ are the normal maps from closed topological 4-manifolds to X up to normal bordism. This can also be described as $[X, K(\mathbb{Z}/2\mathbb{Z}, 2) \times K(\mathbb{Z}, 4)]$. When we talk about exactness of the surgery sequence, we obviously mean exactness at $\mathcal{N}$. We just zoomed in a specific place in the sequence, it continues for a while. This (in all its generality) is arguably one of the most powerful tools in manifold topology.

The knot stuff helps us with the geometric interpretation of showing that $Ker \subset Im$. Let $L \subset S^3$, let $M_L = \partial X_L$ where X_L link trace. Note that M_L is the result of 0-framed Dehn surgery on S^3 along L.

Theorem 6. *(Folklore) Let* $L \subset S^3$ *be a link. It is topologically slice if and only if* $M_L = \partial W^4$ *where* W *is connected, compact (topological* 4 *manifold) such that* $H_1(M_L) \xrightarrow{\iota_*} H_1(W)$ *is an isomoprhism,* $\pi_1(W)$ *is normally generated by meridians of* L *and* $H_2(W) = 0$.

Proof sketch that ever good boundary link is slice implies that every surrgery sequence is exact (i.e for all fundamental groups)

Proof. Let $g : N \to X$ be a degree one normal map, to show the desired result, it suffices to arrange isomoprhisms on π_1 and π_2.
We can "do surgery on circles" on N to change g to a π_1-isomorphism, because the assumptions already imply surgery.
Now let's work on π_2, we already have a surjectivity by degree 1ness, so we need to kill the surgery kernel $K_2(g) = ker(\pi_2(g))$. We need to use the assumption that the image in L_4 is trivial, we do this by viewing $K_2(g)$ as a $\pi_1(X)$ module, which we can do by π_1 isomorphism, and then some restriction stuff to make it a non singular quadratic form, which is stably hyperbolic by assumption of being trivial in L_4.
This for mysterious reasons implies that generators of $K_2(f)$ can be represented by generically immersed spheres which have some properties with respect to the forms.
Now there is some subtlety that in dimension 4 the algebra and geometry are different enough that the obvious idea won't work (whatever that is), but instead we get some insane knots which were drawn on the board. These insane knots come form Kirby calculus. We use this insane knot to construct a four manifold in N, which we will remove, and then perform 0 surgery on a good boundary link, which by the Folklore theorem we can glue back an even nicer 4-manifold. $\square$

4 Talks

4.1 SOFÍA MARLASCA APARICIO: Ultrasolid homotopical algebra

You can think of condensed R-modules as a way to make categories of ”topological modules” abelian (because usually they aren't). Now let S be a **profinite?** space, we can form the free condensed k-module $K[S]^{\square} = \varprojlim K[S_i]$. As opposed to just the free module $K[S]$.
A condensed module M is solid if for every profinite S we have

$$\mathrm{Mor}(K[M]^{\square}, M) \cong Hom(K[S], M)$$

Theorem 7. *If $K = \mathbb{Q}, \mathbb{F}_p$, then solid modules form an abelian subcategory closed under (co)limits and has a compact projective generator $\prod_I K$.*

Definition 8. *A profinite vector space is a cector space of the form $\prod_I K$ and we set $Mor(\prod_I K, K) = \bigoplus_I K$.*

We write $Pro(Vect_K^{\omega})$ for the category of profinite K-vector spaces. A module is ultrasolid is it lies in the sifted cocompletion of $Pro(Vect_K^{\omega})$ (i.e closure under filtered colimits and reflexive coequalizers, which because we are in an abelian setting is the same as closure under sums and cokernels).

We call this category (of ultrasolid K-modules) $Solid_K^{\heartsuit}$ and denote sifted cocompletion by P_{Σ_0}. Now we will be able to define Ultrasolid K-algebras if have a symmetric monoidal structure. On $Proj(Vect_K^{\omega})$ this can be done by $\prod_I K \otimes \prod_J K = \prod_{I \times J} K$

Definition 9. *An ultrasolid K-algebra is a commutative ring object in $Solid_K^{\heartsuit}$.*

Definition 10. *An augmented ultrasolid K.algebra is a an ultrasolid K-algebra R equiped with a map $R \to K$ which is a retract of the inclusion $K \to R$.*
Further we call it complete if $R \cong \varinjlim R/m^n$ where m is the augmentation ideal. It is profinite if the underlying ultrasolid module is a profinite vector space.

Theorem 8. *(Ultrasolid Nakayama) Let R be a complete ultrasolid K-algebra, let M be a profinite R-module, and m the augmentation ideal, then*

$$M/mM = 0 \Rightarrow M = 0$$

There is a choice between derived (simplicial rings) and spectral algebraic geometry (connective spectra), we are going to do derived.

Now we change of topics, and talk about deformation theory for some reason.
Let X be a proper smooth variety over K, and A be a some K algbera with residue filed K (with other assumptions), then a deformation of X over A is a flat map $\tilde{X} \to \mathrm{Spec}(A)$ such that we have a pullback square:

1. X in the top left
2. $\tilde{X}$ in the top right
3. **iel a effacé trop vite...**

A formal moduli problem is a functor $X : CAlg_{smthelse}^{smth} \to S$ where S is **smth** such that $X(*) \simeq *$ and it preserves certain pullbacks.

And then... they end the talk on some theorem, there was some PDF for the talk which I will download.

4.2 TONY MBAMBU KAKONA: Differential bundles in tangent infinity categories

1. tangent categories
2. differential bundles
3. classification
4. Smth else (infinity category setting?)

A tangent category aims to generalize manifolds to other categories. There is a bunch of stuff in the category of smooth manifolds, we will define a "tangent category" to be a category X with:

1. A functor $T : X \to X$ which acts like taking the tangent bundle
2. Nat transformation $p : T \to Id$
3. And a bunch of other stuff which can be guessed from properties of the tangent bundle

Tangent categories are a general framework for geometric settings, categories in differential geometry, algebraic geometry, etc. are all tangent categories. They also appear in abelian categories, where there is some notion where tangent categories capture a bit of functor calculus.

Example 1. *Non geometric example, let $\mathbb{N}^\bullet$ be the category of free $\mathbb{N}$ modules, then Tangent structure is given by $T(\mathbb{N}^k) = \mathbb{N}^k \times \mathbb{N}^k$.*

There is a natural notion of functor of tangent categories, called "lax tangent functor", which is to say $F : (X, T_X) \to (Y, T_Y)$ and a natural transformation $\alpha : FT_X \to T_Y F$ which makes everything commute, there is also a notion of natural transformation of lax tangent functors.

Okay now part 2, in a tangent category we can define a tangent bundle E over M, which has some properties. This abstract characterization (which I did not specify) is a good notion because the tangent bundles in the tangent category of smooth manifolds is exactly the vector bundles (as desired). A differential object is a differential bundle over the terminal object.

We call a functor F lax differential if $T^n \circ F$ preserves pullbacks over the terminal object and the map $\alpha : F \circ T_X \to T_Y \circ F$ is cartesian, i.e the naturality squares are pullbacks. We call F strong differential if further it preserves the terminal object.

There is a good example of a differential bundle in the category $\mathbb{N}^\bullet$ which is the projection $\mathbb{N} \to 0$. This is indeed a good example by

Proposition 2. *Differential bundles in any differential category X are equivalent to lax differential functors $F : \mathbb{N}^\bullet \to X$.*

This theorem can be used as a definition of differential bundles in the infiinity category cases if we can replace $\mathbb{N}^\bullet$ by an approriate infinity categorical replacement. It turns out $E = N_\bullet(Span(FinSet))$ is a good replacement for this as $Fun(\mathbb{N}^\bullet, C) \cong Mon(C)$ and $Fun(E, C)$ is equivalent to E_∞-monoids in the infinity category C.

4.3 SHAI KEIDAR: Higher Galois Theory

So recall some stuff about classical galois theory. But rephrase the definition to be better for generalization

Definition 11. *A field F is a G extension of E if we have an inclusion $E \to F$ and a G action on F such that*

1. $F^G \cong E$
2. $F \otimes_E F \cong \prod_G F$

With this definition we can really nicely generlize, let C be a symmetric monoidal category. And $R \in CAlg(C)$ and G a finite group. Then

Definition 12. *A $S \in CAlg(C)$ is a G extension of R if we have a map $R \to S$ and a G action on F such that*

1. $S^{hG} \cong R$
2. $S \otimes_R S \cong \prod_G S$

One example of where we could do Galois theory is with spectra where we have pretty cool theorems (also cool thing, these ideas come from Rognes).

Definition 13. *We call $R \in CAlg(Sp)$ is connected if $\pi_0(R)$ is connected as an algebra*

Consider Pr^L the category of presentable categories, this comes equipped with the (Lurie) tensor product.

We have classical Galois descent, which is to say if $E \subset F$ is a G extension, then $Vect_F^{hG} \cong Vect_E$

We call an $\mathbb{E}_1$ space m finite if it is m-trucated and all non trivial homotopy groups are finite.

Why do we introduce this, as an example of a different finiteness notion to work. We don't actually need to work with finite groups to have a galois theory for an arbitraty for a category C, we need to work with groups that C considers small, and this can vary wildly depending on C (sometimes it even includes less than all finite groups).

Definition 14. *A category is m-semi additive, if colimits and limits over m finite spaces agree. We say infinity semi additive if semi additive for all m*

Many categories are ∞-semi additive. This is enough for finite groups to be small enough to do Galois theory (I think).

He is talking a bit fast for me to understand much... and definitely too fast to write down enough for these notes to be rly useful.

He wrote at some point $Vec_{\mathbb{C}}^{(n)}$ which is the n-fold categorification of $Vec_{\mathbb{C}}$, I am a bit unclear on what this means.

Example of a result, the galois closure of $Vect_{\mathbb{C}}$ is the category of C_2-graded vector spaces (or super vector spaces).

There is some "Tio" guy who seems to work with stuff like n-extended TQFT, which are functors from the infinity category $Bord_n \to \overline{Vect}_{\mathbb{C}}^{(n)}$ the algebraic closure of the category $Vec_{\mathbb{C}}^{(n)}$. Which relates generalize Galois theory and TQFT, sounds interesting.

4.4 MAX BLANS: On the chain rule in Goodwillie calculus

So what is it? In one sentence it is a theory for approximating functors in homotopy theory with easier functors.

Let $F : Top_* \to Top_*$ be a functor, then Goodwillie calculus gives a tower $\cdots \to P_nF \to P_{n-1}F \to \cdots$ which can approximate F, such that under good circumstances the inverse limit of the tower is F. The intuition is that it interpolates between the stable and unstable part. We understand the fiber of $P_nF(X) \to P_{n-1}F(X)$ pretty well, as the infinte delooping of a certain spectra $\Omega^\infty(\partial_n F \otimes \Sigma^\infty X^{smth})_{smth}$.
The important part is the $\partial_n F$, called the derivatives of F. These fit into a symmetric sequence (same notion as in the theory of operads)

$$(\partial_1, \partial_2, ...)$$

The category of symmetric sequences is monoidal under the composition product $\circ$. The algebras in the category of symmetric sequences of spaces are the operads on spaces.
The natural question, given that the derivatives fit into a symmetric sequence, is whether in fact they form an Operad.

Theorem 9. *(Ching 2005) The derivatives of the identity $\partial_* Id_{Sp}$ is the Lie operad in Sp.*

An example of why we care about the lie operad

Theorem 10. *(Heurts 2018) The category of v_n periodic spaces is equivalent to $Alg_{Lie}(Sp_{T(n)})$*

Arone and Ching constructed for every $F : Top_* \to Top_*$ a $\partial_* Id_{Top_*}$-bimodule structure on $\partial_*(F)$. (where the algebra structure is given by Ching's 2005 theorem).
Then using this we can formulate

Theorem 11. *(Chain rule) We have the following chain rule for Goodwillie calculus*

$$\partial_*(FG) = \partial_*(F) \circ_{\partial_*(Id)} \partial_*(G).$$

There is a conjecture to generalize this which is that for a category C which has a Goodwillie calculus. then

$$\partial_* : Fun_n(C, C) \to SymSeq(Sp(C))$$

is lax monoidal. By the subscript $(-)_n$ we mean functors nice enough to admit the Goodwillie calculus and $Sp(C)$ is the "stabilization". This conjecture implies many nice things because Lax-monoidal functors preserves algebras and modules, and the identity is clearly an algebra in the domain and every functor is clearly a module over the identity. Thus when this conjecture holds, we get some of the above results "for free".

This conjecture has been shown to hold for "presentable differentiable categories", and in fact in this case we have the chain rule in the sense that the natural map $\partial_*(F) \circ_{\partial_*(Id)} \partial_*(G) \to \partial_*(FG)$ is an equivalence.

In fact this result was then upgraded to a more infinity categorical language.

4.5 KAIF HILMAN: Parametrised Functor Calculus and a universal property of Mackey Functors

Alternative title: Equivariant to Goodwillie dictionary.

"Review" of Goodwillie calculus: Let $F : C \to D$ be a functor between "stable" categories which is "reduced" (i.e preserves the 0 object). Then we have a tower $\cdots \to P_nF \to P_{n-1}F \to \cdots$ such that ...

Detection (cross-effect): Given a functor as above, then we have a functor $cr_dF : C^d \to D$. Then we have a trick to measure exscivieness (i.e for what n we have $F \simeq P_nF$ is an isomorphism). The tool is that if a functor is n-excisive and $cr_n(F) = 0$, then it is actually $n+1$ excisive.

Classification: In some cases (**All??**) we can take the fiber $D_dF = Fib(P_dF \to P_{d-1}F)$ and in some cases we know what this fiber is.

The goal is to establish a dictioary between equivariant homotopy theory (with finite groups) and goodwillie calculus.

1. Genuine G spectra $Sp_G \simeq Mack(O_G, Sp)$. The category of d-excisive endofunctors of spectra corresponds to Mackey functors from the category of finite sets with morphisms surjective up to d elements into spectra.
2. Geometric fix edpoints: In the case $G = C_p$, we have a pullback square, with top left X^{C_p}, top right $X^{\Phi C_p}$, bottom left X^{hC_p} and bottom right X^{tC_p}. We can recover the taylor tower of F just from the symmetric sequence of the derivatives.

The category of spectra is the initial **smth smth** and categories of Mackey functors is initial in **smth smth else**...

We use this idea (which I didn't really understand) to construct a category $\underline{Sp}_d$ which is a "gadget" category used to construct the equivalence. It is a d-"parameterized" categories, which means we have one category for each integer $1 \leq r \leq d$. The point is that if we can construct a category carefully, we can get a map for free by the universal property, so we just have to show it is an equivalence, which will be done by induction, which makes sense because we are in a "parameterized" world.

4.6 Milicia Jovanovic: Cohomology operations on Polyhedral products

We want to define the polyhedral product, let K be a simplicial complex with m vertices and let (X_i, A_i) be m topological pairs. WE define $(X_i, A_i)^K$ to be the polyhedral product as the colim over $\sigma \in K$ of $(X_i, A_i)^\sigma = \prod Y_i$ with $Y_i = A_i$ when $i \notin \sigma$ and X_i if $i \in \sigma$. (I think unsure I was able to copy correclty)

If we let K to be disjoint union of m vertices, the polyhedral product gices the wedge of the X_i and if $K = \Delta^m$ we get the product of the X_i, so polyhedral product interpolates between these. In the case $(X_i, A_i) = (D^2, S^1)$ for all i we speak of a "moment angle complex". Because for moment angle complexes the only choice is the simplicial complexx, so we get a functor $Ob(SimCpx) \to Top$.

Recall Steenrod squares, written $Sq^k : H^n(-, \mathbb{Z}/2\mathbb{Z}) \to H^{n+k}(-, \mathbb{Z}/2\mathbb{Z})$. They assemble into the Steenrod algebra.

Okay now lets try and work on some simple cases:

1. Davis Januskiewicz space $DJ_K = (\mathbb{C}P^\infty, *)^K$. (Did not write out the details of the example)
2. In general we can understand the action on polyhedral products of the form $(X, *)^K$, i.e with the pair's constant equal to a pair $(X, *)$.
3. For moment angle complexes we can understand the cohomology as a tor-algebra. One particular study which can be done is that if Z_K is the moment angle complex of K, then ΣZ_K splits, and we can kinda study the Steenrod action from this.
4. We can apply the above idea to $K = P_6^2$ a minimal decomposition into a simplicial complex of $\mathbb{R}P^2$ which is in some sense the minimal space with non trivial Steenrod action.

In the definition of polyhedral product, we can replace the product by other stuff, and still get smth interesting?

At the end of the talk, they discussed how non trivial Steenrod action can propagate to polyhedral products and related stuff.

4.7 VICTOR SAUNIER: Trace methods for stable categories

Joint work with Y. Harpaz and T. Nikolaus

Motivation (?) from linear algebra: Let A be a real matrix, sps we want to know it's determinant. Well this can be quite hard. But In some cases we can do nice stuff, i.e $det(I+\epsilon A) = det(I)+\epsilon tr(A)+o(\epsilon)$.
This follows from several different ways, the speaker gave a proof using:

1. that the trace is uniquely characterized by
   (a) linearity
   (b) $tr(AB) = tr(BA)$
   (c) $tr(E_{11}) = 1$
2. $det(I + AB) = det(I + BA)$
3. now show that the linear part of $det(I + \epsilon A)$ satisfies the defining properties of the trace.

In fact more generally we have a nice interaction of the two $det \circ exp = exp \circ tr$ and something similar.

In this talk, let's use the perspective that K-theory is a functor from the infinity category of stable infinity categories to spectra: $K : Stable \to Sp$. The goal of the talk is to convince that K-theory is a "determinant" for stable infinity categories.

Definition 15. *Let $F : C \to D$ be a functor of stable infinity categories, we define $C \oplus_F D$ to be the pullback of $C \xrightarrow{F} D \to D \xleftarrow{cod} D^{\Delta^1}$.*

Definition 16. *A functor of stable infinity categories $F : C \to D$ is called additive/splitting if $Stable \to Sp$ sends $C \oplus_F D$ to the direct sum of the spectra associated to C and D.*

Theorem 12. *There is a natural transformation $\Sigma^\infty \circ |-| \circ Core \to K$ which is initial in the category of splitting funtctors under $\Sigma^\infty \circ |-| \circ Core$.*

If C has finite limits and it is pointed, then it can be stablized by taking an inverse limit of $\cdots \xrightarrow{\Omega} C \xrightarrow{\Omega} C \xrightarrow{\Omega} \cdots$, call it $Sp(C)$. From Goodwillie calculus we have the intution that stable stuff is in some sense linear, so linear approximation at a point should kinda be "tangent" stuff

Definition 17. *We define $T_X C = Sp(C_{X//X})$, these can assemble together into a "tangent" bundle $TC \to C$*

Now natural question in light of the goal mentionned above, what is tangent stuff in the category of stable infinity categories?

Theorem 13. *(Someone-Nikolaus-S.)* $T_C\,\text{Stable} \cong Fun^{smth}(C^{op} \otimes C, Sp)$

Then he started talking really fast, so I wasn't really able to understand, but in essence he defined a bunch of stuff to state

Theorem 14. *The linearization of K^{cyc} in M is $THH(C, M) = tr(M)$*

This seems really interesting, I really would like to understand more about what he said, because it all seems very interesting, but it went a bit over my head. Maybe search up Harpaz, Saunier and Nikolause on Arxiv

4.8 BRANKO JURAN: The algebraic K-theory of algebraic tori via equivariant homotopy theory

The plan is to explain the words, joint work with Conrad, Riebel and Bai (no idea if I spelt the name right).

Algebraic K-theory:

Definition 18. *Let R be a commutative ring, then consider $P(R)$ the category of finitely generated projective modules. Then $K_0(R) = \pi_0(P(R))^{Gr}$.*

Definition 19. *Let $K(R) = BIso(P(R))^{gr}$, this is a connective spectra, whose homotopy groups are the K-groups*

Remark 15. *Remark, K-theory is defined for schemes as well (Quillen 1973).*

There are some properties (assuming Noetherian and regular)

1. (homotopy invariance) i.e $K(R) \to K(R[t])$ is a weak equivalence.
2. (Localizing) Let X be a scheme and $U \subset X$ an open subscheme, then we have a fiber sequence

$$K(X \backslash U) \to K(X) \to K(U)$$

One example computation using only these two properties is that $K(F[t, t^{-1}]) = K(F) \oplus \Sigma K(F)$.

Okay nice little intro, let's talk about algebraic Tori.

Definition 20. *An algebraic torus over a field F is a group scheme T defined over $Spec(F)$, such that $T_{\bar{F}} \cong Spec(F[t, t^{-1}])^n$.*

Definition 21. *A topological torus is a connected commutative compact Lie group.*

Proposition 3. *The category of topological toris is equivalent to the category of finitely generated free abelian groups. By the map that sends a torus T to $Map_{Lie}(S^1, T)$.*

Proposition 4. *The opposite category of algebraic tori over F is equivalent to the category of finitely generated generated abelian group equipped with an action of $Gal(F)$ the absolute Galois group. This map sends T to $Map(T_{\bar{F}}, Spec(\bar{F}[t, t^{-1}]))$. This map is called Λ^*.*

Combining the above two result we get that algebraic tori over F are equivalent to topological tori with an action of $Gal(F)$.

Now we move on to "the result". Recall $K(Spec(F[t, t^{-1}])^n) = K(F)[(S^1)^n]$. Now for X an F-scheme, we have an equivariant spectra $\underline{K(X)} \in Sp^{Gal(F)}$.

Theorem 16. *Let T be an algebraic torus over F, thn there exists an equivalence*

$$\underline{K(T)} \cong \underline{K(F)}[B\Lambda^* T]$$

4.9 NINGCHUAN ZHENG: Equivariant algebraic Ktheory and Artin L-functions

The first part is about algebraic K-theory and zeta functions, the second part is about twisted Quillen-Lichtenbaum conjectures.

What are the Riemann and Dedekind Zeta function. Let F be a number field. The Dedekind zeta function is

$$\zeta_F(s) = \sum_{0 \neq I \subset \mathcal{O}_F} \frac{1}{(\#O_F/I)^s}$$

Taking $F = \mathbb{Q}$ yields the classical Riemann zeta function, and properties of ζ_F look a lot like those of ζ.

The Dirichlet unit theorem and class number formula can both be formulated as special values or numbers related to ζ_F. In particular

$$ord_{s=0}\zeta_F(s) = rank_{\mathbb{Z}}(\mathcal{O}_F^{\times}) = rank_{\mathbb{Z}}(K_1(\mathcal{O}_F))$$

And

$$\zeta_F^*(0) = \text{classnumber}$$

(and this one can also be related to some K theory thing)
We see some connection between Dedekind functions and K-theory.

For finite fields, we also have some nice stuff. Define

$$\zeta_F(\mathbb{F}_p, s) = \frac{1}{1-p^{-s}}$$

The K theory of $\mathbb{F}_p$ is fully known, and so one can more or less directly "notice" a version of the Quillen Lichten-Baum conjecture.

Then there is a slide mentioning the result of Borel which computes the rank of K-groups of integer rings of number fields. And then mentionning the Quillen-Lichtenbaum conjecture.

Now the Quillen Lichtenbaum conjecture is nice, but in number theory we have Dedekind zeta functions which are twisted by Galois representations, called Artin L-functions, can we obtain a "twisted" version of Algebraic K-theory such that we get a "twisted" Quillen-Lichtenbaum conjecture. And this guy did some of that.

Okay so... equivariant algebraic K-theory. There is a theorem which says that the equivariant algebraic K-theory of G-Galois cover of schemes admit a genuine G-spectrum structure. And then it gets really complicated.

4.10 VALENTINA ZAPATA: CASTRO Monoidal (infty, n)-categories

Recall: a category is ...

An (∞, ∞) category is smth with 0-morphisms, 1-morphisms, ..., n-morphisms, ...
An (∞, n) category is an (∞, ∞) category where all the k-morphisms are invertible for $k > n$.
A general persepctive is that (∞, n) categories are enriched over $(\infty, n-1)$ categories.

Recall: a simplicial set is...

Recall: A bisimplicial set (or simplicial space) is a ...

There is a model category structure on the category of bisimplicial sets denoted CSS (complete segal space) which is good for modeling $(\infty, 1)$ categories.

Definition 22. *Let C be a small category, we define θC to be the category with objects pairs $([m], (c_1, ..., c_m))$ ($[m]$ an ordered set) and morphisms are given by $(\delta, \{f_{ij}\})$ where $\delta \in Fun([m], [n])$ and $f_{ij} : c_i \to d_j$, but only defined for i, j such that $\delta(i-1) < j \leq \delta(i)$.*

Definition 23. *We define $\theta_0 : 1$ and then define inductively $\theta_{n+1} = \theta\theta_n$.*

Definition 24. *We define a θ_n-space to be a contravariant functor from θ_n to $sSets$. These assemble into a category.*

Theorem 17. *(Rezk) There is a model structure on θ_n-spaces which is a good model for (∞, n) categories.*

Now we can define monoidal (∞, n) categories. We do this by "generalizing" the definition of a monoid as a category with one object.

Definition 25. *An (∞, n)-category with a monoidal structure is an $(\infty, n+1)$ category with one object.*

4.11 SVEN VAN NIGTEVECHT: A synthetic version of topological modular forms

The goal is to resolve a circularity in the computation of some spectral sequence for TMF.
Why care? A lot of understanding of the stable stems comes from TMF.

Question: How do we understand $\pi_n(\mathbb{S})$ for large n.
Answer: Test it against other spaces.

Question: Given a map $f : \mathbb{S} \to X$ how do we understand $\pi_i(f)(a)$ for some $a \in \pi_i(\mathbb{S})$.
Answer: For example with functorial spectral sequences which compute homotopy group. i.e the Adams Novikov spectral sequence (abbreviate to ANSS).

We will test the sphere spectrum against K-theory and topological modular forms.

So we use the ANSS for spectra representing K-theory/TMF, and then after understanding it, we use a (nice) map $\mathbb{S} \to KO$ (or some other). This map will give a map of spectral sequences, so we can use our understanding of the ANSS of KO to "detect" information in the ANSS of the sphere (and thus let's recall we detect information of the stable stems).

There are three versions of TMF. tmf is connective, TMF is periodic, Tmf is neither.

In some specific case, this strategy doesn't act like one would hope, so we work with synthetic spectra instead,

What are synthetic spectra? E-Synthetic spectra are to E-Adams spectral sequences what spectra are to homology.
The infinity category of synthetic spectra Syn_E is amazing, like it has all the nice properties we would want it to have.
It comes with a couple functors, $\nu : Sp \to Syn_E$ called synthetic analog, a functor $Syn_E \to SSeq$ (the category of spectral sequences) such that precomposing with ν yields the Adams spectral sequence. And a third functor $p : Syn_E \to Sp$.

There is a "Synthetic" analog of Tmf called "Smf".

4.12 Kaelyn Willingham: Spectral properties of the algebraic path problem

Plan

1. The algberaic path problem
2. Motivation: link prediction
3. Cellular Sheaf theory
4. The Sheaf laplacian

What is the algebraic path problem, generalizes path problems on graphs under a single construct. E.g the shortest path problem or ”transition probability problem”.
(He gave an example of a graph and it seems to imply that our graphs are directed)

Determining the shortest path algorithmically is not an easy thing to do, but we have methods:

1. Dikjstra's algorithm
2. Bellman Ford Algorithm
3. A^* search algorithm
4. Floyd Warshall algorithm
5. Johnson's algorithm.
6. Another one which he changed the slides too fast

The Bellman Ford algorithm is mathematically interesting bc it allows negative edge length, but: it isn't made to be efficient, and it can run into ”runtime” errors if you have an edge which is a self loop of negative length (bc then to minimize length, the algorithm wants to go around forever to send the length to negative infinity).

This algorithm can be generalized to any $\sim$ ring things, seen as a graph? Okay I didn't really understand the equation.

The transition probability problem is take a graph where length of edges is 0 or 1, consider a random walk on this graph, then we can ask ”what is the likelihood to go from v_1 to v_2”, and this can be solved by phrasing it as a Markov chain. An algorithm which solves this problem is the google page rank problem.

A link prediction problem, is given a collection of vertices, and a random graph, what is the probability that two given vertices are connected.

A generalization of all the above problem, is to take graphs in the category of vector spaces.

Then for some reason a natural question is to interpret ”higher dimensional” algebraic path problems in terms of cellular sheaves, at which point we have access to homology theories and stuff.

A cellular sheave in a category D on a cell complex Ω is a covariant functor $F : P_\Omega \to D$.

4.13 ZACHARY GARDNER: Moduli of truncated prismatic (G, mu)- displays

I missed the beginning of the talk due to delay in the other talk.

Heuristic: p-divisible groups capture the p-local pieces of abelian schemes.
We fix n the level, h the height and l the dimension.

Theorem 18. *The category of p divisble groups is equivalent to the direct limit of level n p-divisble groups.*

The category of p divisible level n groups is a smooth Artin p-adic formal stack.

The category of p-divisible level n + ***some other property*** *is a smooth, quasi-compact, 0-dimensional, p-adic Artin Formal stack.*

The functor $p - div^{Smth}_{n+1} \to p - div^{Smth}_{n}$ *is smooth and surjective.*

Suppose F is a perfect field of characteristic p, then we can associate to it $W(F)$, the ring of Witt vectors, which we should view as the "p-adic power eries w/ coeff in F".
For example $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ (the p-adic integers)

Definition 26. *A Dieudonné module over F is a morphism is a pair* (G, E)*, where* $G : E \to E$ *is a morphism in* $Vect(W(\mathbb{F}_p))$ *and we require* G *to be* ϕ*-linear, where* ϕ *is the Frobenius map.*

This construction is useful, e.g because

Theorem 19. *The p-divisble groups are isomorphic as a category to the category of Dieudonné modules over* $\mathbb{F}_p$.

The goal/dream/idea is to generalize the above result, as a first step to perfect $\mathbb{F}_p$-algebras.

Okay now we will give a field tour of "prizmatisation"
The prismatisation of a ring R^P, is a formal p-adic stack associated to R.
There is a frobenius map $R^P \to R^P$.
There is something called a De Rham point, which is a map from $SpfR$ to R^P

In the perfect case we have $R^P = Spf(W(R))$

A filtered prismatisation R^N is a filtered stack.

In the perfect case, this corresponds to giving $W(R)$ the p-adic filtration.

Idea: replace p-divisible group of level n ny vector bundles over an appropriate space inside of R^N.
You can think of R^N as the freeest way to make R geometric.

Snytomification of R is a thing which they mentioned. Define: $BT_n(R) = Vect^{[0,1]}(R^{syn} \times_{Spec/\mathbb{Z}/p\mathbb{Z}} Spect/\mathbb{Z}/p^n\mathbb{Z})$. I have no idea what this talk is about.

4.14 KAMEN PAVLOV On the homotopy classification of 4-manifolds with fundamental group $\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

All manifolds are topological, connected, closed, oriented.

Definition 27. *For a space X, the Postnikov 2-type of X, written $P_2(X)$ is the 2-truncation $p_2^X; X \to P_2(X)$- This is determined by the k-invariant $k_X \in H^3(\pi_1(X), \pi_2(X))$.*

Theorem 20. *(Baues-Bleile, smbdys) Given two manifolds M, M'. with the same Postnikov 2-types. Then if $H_4(p_2^M)([M]) = H_4(p_2^{M'})([M'])$, then in fact M and M' are homotopy equivalent.*

Recall the classical intersection form is the map $I : H_2(M) \times H_2(M) \to \mathbb{Z}$ which sends (A, B) to their intersection number.

Definition 28. *Equivariant intersection form, is a map $H_2(M^u) \times H_2(M^u) \to \mathbb{Z}[\pi_1]$. Which sends (A, B) to $\sum_{g \in \pi_1} I(Ag^{-1}, B)g$.*
Dually, it is a map $\lambda_M : H^2(M, \mathbb{Z}[\pi]) \times H^2(M, \mathbb{Z}[\pi]) \to \mathbb{Z}[\pi]$ given by $< \beta, \alpha \cap [M] >$.

We have a neat way to bring this down to the Postnikov 2-type of M, by a map $H_4(P_2(M)) \to \mathrm{Mor}(H^2(P_2(M), \mathbb{Z}[\pi]), \mathbb{Z}[\pi])$. Which is defined in some way.

Definition 29. *A quadratic 2-type is the following collection of data $Q(M) = (\pi_1(M), \pi_2(M), k_M, \lambda_M)$*

It was conjectured/hoped this is a complete set of invariants for homotopy types of Manifolds. We know this for some groups, e.g

1. finite cyclic
2. finite abelian 2-groups
3. finite dihedral
4. $\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

But it in fact does not hold in general, for example it fails for $\pi_1 = \mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ which can be seen by considering $L(5, 1) \times S^1$ and $L(5, 2) \times S^1$. Now the conjecture only stands for finite groups.

Theorem 21. *(Kasparovich-P) For the fundamental groups $\pi_1 = \mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ where p is prime, then for any given quadratic 2-type, there are at most p-homotopy types.*

One tool was

Theorem 22. *Given the cannonical map $c : M \to K(\pi_1(M), 1)$, the class $c_*([M]) \in H_4(\pi_1(M), \mathbb{Z})/Aut(\pi_1(M))$ determines the $\pi_2(M)$ stably.*

4.15 EMMA BRINK: Equivariant bordism and Thom spectra

Theorem 23. *(Thom, Pontryagin) θ-oriented bordism is represented by the Thom spectrum $m\theta$*

Let M be a smooth manifold with a map $M \to bO$.

We have $O^\theta(M) = \pi_*(smth)$. By varying θ we get different bordism notions.

Bordism is a cohomology, so for a manifold with boundary we get a map $O^\theta(M) \to O^\theta(\partial M)$. We have $M_n^\theta(X) = smth$ this smth is cobordism classes of θ-oriented manifolds over X.

We have a functor $Fun(Top/bO, Sp) \to Fun(bO, Sp)$, we have a J homomoprhism $bO \xrightarrow{J} Pic(Sp) \to Sp$ (she writes really small, so I kinda have no idea what is happening... :( )

Anyway this is all classic, to make it all equivariant we need to replace bO with aG-equivariant analog. She does it for G compact Lie. And the construction is done by taking the spectra representing the cohomology which sends X to the G-bundles over X. Everything is made equivariant by taking... $Fun(O(G)^{op}, Cat_\infty)$... maybe? no idea.

Something about parametrized homootpy theory which does something for equivariant theory. Yeah I sadly will stop taking notes because I am unable to understand/concentrate/follow.

4.16 ALICE ROLF: Endomorphisms and Automorphisms of the Framed Little Disk operad

Let's spoil le twist: the endomoprhism will be exactly the automorphisms.

Examples of operads:

1. The little discs operad E_d, where the n space is the space of affine embeddings $\sqcup_n D^d \to D^d$. There is a natural action of S_n on $E_d(n)$.
2. Framed little disc operad E_d^{so}, it is almost the same as E_d, only we allow rotating when embedding the discs.

Up to homotopy $E_d(n) \simeq conf_n(\mathbb{R}^d)$. In particular $E_d(2) \simeq S^{d-1}$.

Definition 30. *A map of operads $f : O \to P$ is a collection of $\mathbb{N}$ maps $f_n : O(n) \to P(n)$ which are S_n equivariant, and such that these maps respect the approriate notion of "compositions"*

And so now, question: what is $Aut(E_d)$ (homotopy automorphisms)? It is quite mysterious.
We know: $O(d)$ acts on E_d interestingly.
We also know the following suprising fact: H $Homeo(\mathbb{R}^d)$ acts interestingly on E_d.

Let's draw some parralels

1. The groups $homeo(\mathbb{R}^d)$ admits a stabilisation morphism $homeo(\mathbb{R}^d) \to homeo(\mathbb{R}^{d+1})$.
2. Local automorphisms of manifolds
3. We have an inclusion $Homeo(\mathbb{R}^d) \to Emb(\mathbb{R}^d)$ which is a homotopy equivalence by Kister's theorem.

1. The groups E_d admits a stabilisation morphism $Aut(E_d) \to Aut(E_{d+1})$.
2. Local automorphisms of "configurations in manifolds"
3. We have an inclusion $Aut(E_d) \to End(E_d)$ which is a homotopy equivalence by a theorem of Horel, Krannich and Kupers.

Theorem 24. *(by the speaker) The inclusion $Aut(E_d^{so}) \to End(E_d^{so})$ is an equivalence*

Some ideas of the proof:

1. Endomorophisms of E_d^{so} correspond to squares in Top commuting up to (specified) homotopy with the top row an endomorphism of $BSO(d)$ and the bottom row the identity of $BAut(E_d)$
2. We have a restriction map to the aritty 2 case $BAut(E_d) \to BAut(S^{d-1})$
3. There is a result which says that $U \times End(SO(2n+1))/Inn(SO(2n+1)) \to [BSO(2n+1), BSO(2n+1)]$ is a weak equivalence where U is some subset of the integers.

4.17 JESSE COHEN: Bordered Floer theory, Hoschild homology, and links in $S^1 \times S^2$

The main result is there is a spectral sequence whose E^2 page is the reduced Khovanov homology of the mirror of a link and whose E^∞ page is the Heegard Floer homology of the branched double cover of L in S^3.

And the above result is well known, he established a generalization.

What is Heegard Floer Homology? it is a functorial invariant of 3-manifolds and cobordisms between them.

It has a wide range of applications in low dimensional topoogy: e.g detecting manifolds fibered over S^1, exotic structures on manifolds, $\mathbb{Z}^\infty$ summand in the 3-bordism group

A Heegard diagram for a three manifold M is a choice of a closed surface of genus g, i.e Σ_g, and two collections of pairwise disjoint curves in Σ_g, say α and β.

Given this, we can construct $\Sigma_g^g/(\alpha \cup \beta)$, this can be given the structure of a Kähler manifold, and we se it to define the Heegard floer homology of M.

Khovanov homology is a combinatorially defined invariant of links and link cobordisms. We take a link diagram, label them and form the cube of resolution by replacing resolutions by the "formal bigraded complex". He is talking so fast oh my god. After doing that apply a 2-dimensional TQFT, the one associated to the commutative frobenius algbera $\mathbb{F}[x]/x^2$. Homology of this is an invariant of links, which is functorial with respect to link cobordisms.

It detects a bunch of stuff about links and exotic stuff in dimension 4.

These are hard to compute. So there is a thing called "bordered Floer homology" is an invariant of 3-manifold with boundary which is better than the non bordered case? Similarly to Heegard floer homology, this admits a definition via bordered Heegard diagrams. The definition seems super hard.

The key ingredient in the speaker's result is some technical result relating the chain complexes defining the homology theories we just mentionned. Saying that they are homotopy equivalent under some assumptions.

Let C_n be the set of crossingles matchings on $2n$ points, we can define a thing called a "Khovanov arc algebra" using this idea (no idea how).

He keeps saying "quantum shift" and I have no idea why.

Notice that any link in $S^1 \times S^2$ whose homology class with $\mathbb{Z}$ coefficients is divisible by 2 is the closure of a tangle. Which is interesting bc it says something about one of the homologies we mentioned (I think).

Now I am quite lost I must say...

4.18 MARCO VOLPE: Traces of dualizable categories and functoriality of the Becker-Gottlieb transfers

Let $X \to Y$ be a map of topological spaces, then we have a map in homology $\mathbb{S}[x] \to \mathbb{S}[y]$, obvious so far! Now if f has finitely dominated homotopy fibers, we get a map the "wrong way" $\mathbb{S}[Y] \to \mathbb{S}[X]$, called the transfer map (associated to $X \to Y$), this uses the technology of parametrized spectra **maybe?**.
To make things easier, let's work with even nicer maps, say proper locally contractible, in which case we can construct the transfer map using the technology of sheaves.

Definition 31. *Let X be a locally compact hausdorff space, then $Sh(X, Sp)$ (which we denote $Sh(X)$ from here on out) is the category of contravariant functors $\tau_X \to Sp$ such that for any open cover U_i of $U \subset X$ we have $F(U) = lim F(U_i)$.*

Given a map $f : X \to Y$ by precomposition and contravariance we get a map $f_* : Sh(X) \to Sh(Y)$, now by abstract nonsense, this map has a left adjoint adjoint $f^* : Sh(Y) \to Sh(X)$.

Definition 32. *We say that $f : X \to Y$ is locally contractible if f^* has a left adjont $f_\#$. We call a space locally contractible if the map $! : X \to *$ is locally contractible.*

Example 2. *For X locally contractible we have $!_\#(\mathbb{S}_*) = \mathbb{S}[X]$.*
This justifies viewing $f_\#$ as a "fiberwise homology" (askip).

Then a lemma was mentionned stating that under assumptions $f_\#\mathbb{S}$ is dualizable with dual $f_*(\mathbb{S})$.

With this he defines the transfer map by some rather long composition which I am too lazy to copy.

The question is then whether this transfer map is functorial or not?

Theorem 25. *(Ranzi, Volpe, Wolf) There is a contravariant functor of infinity categories, from the category of locally compacth Hausdorff, locally contractible, with locally contractible proper maps as morphisms to the category of spectra which sends a space to its homology $\mathbb{S}[x]$ and maps to their Becker Gottlieb transfer.*

The proof uses the following lemma

Lemma 2. *For X a compact Hausdorff spaces, $D(THH(Sh(X))) \simeq \mathbb{S}[X]$ where D is the Spanier Whithead dual.*

4.19 LEOR NEUHAUSER: Rigid Algebras are right adjoint to cospans

Motivation: rigid categories. Also Everything is infinity-categorical,

A symmetic monoidal category is a categorification of commutative monoids. Can also be done with presentable stable categories. Archetypal examples have good properties, e.g: Dualizable and compact objects are dualizable.

Definition 33. *A sym monoidal category is rigid if the unit object is compact and the tensor product is internally left adjoint.*

Good ressource; Maxim Ramzi's preprint.

So our notion of symmetric monoidal category is $CAlg(Pr_{st})$ (category of presentably stable categories). We can replace Pr_{st} by some othre category (of categories?), and we can also have rigid noions in this generalized setting.

Recall the definition of the cospan category of a category C denoted by $cospan(C)$. Via the coproduct in C this can be made into a symmetric monoidal category.

In any 2 category we can define a notion of left/right adjoint in perfect analogy with the case of category of categories. (the definition using triangle identities not the other one).

In a sym mon an object is dualizable if there is a dual object $X^\vee$ such that there is an evaluation map $X \otimes X^\vee \to \nVdash$ and a coevaluation map the other way, such that these two maps satisfy some triangle identity thing.

This talk is about these notions in the cospan category.

In any symmetric monoidal 2-category there is a notion of commutative algebra, we call an algebra A rigid if the unit is left adjoint and the multiplication map is left adjoint (in the category of A-bimodules)

A commutative frobenius algebra is an augmented algebra such that $\epsilon \circ \mu$ exhibits A as self dual. Defined another way a Frobenius algebra is an algebra and a coalgebra such that th multiplication and co multiplication satisfy the Frobenius relations.

You can think of rigid algebras as a categorification of Frobenius algebras.

Now apparently every object in C is cannonically a rigid algebra in the category $cospan(C)$ and in fact all the rigid algebras are of this form.

At this point the talk became hard to follow. He was talking about ”6 functor formalism” whatever that is.