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Derivatives of K-Theory

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1 Introduction

1.1 Acknowledgments

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1.2 Extended abstract

Our goal in this project is to trace out a natural next step in applications of Goodwillie calculus to Algebraic K-theory.

Goodwillie calculus is a theory for approximating functors $F : \mathcal{C} \to \mathcal{D}$ between suitably nice homotopical categories (see definition 2.1.2), which was developed by Goodwillie in a series of three landmark papers [13], [14] and [15] in the case where "homotopical" means specifically the category of spaces. And was extended to a large class of ∞-categories by Lurie in chapter 6 of [23]. Specifically Goodwillie calculus provides a suitable definition of degree n functors, the so called n-excisive functors, and then proves the existence of a universal n-excisive approximation P_nF of a functor $F : \mathcal{C} \to \mathcal{D}$ between suitably nice categories. The theory of Goodwillie calculus also defines an nth derivative $\partial_n F$, defined in terms of P_kF, but in practice is easier to compute. So the general strategy laid out by Goodwillie calculus to study a functor consists in first understanding its derivatives, then the n-excisive approximations and only then studies the original functor.
More recent ideas, originally due to Arone and Ching, and as present in this project due to Blans and Blom, suggest that for a functor $F : \mathcal{C} \to \mathcal{D}$, one ought to first study the derivatives of $\Sigma^\infty_{\mathcal{D}} \circ F \circ \Omega^\infty_{\mathcal{C}}$, and then try to deduce information regarding the derivatives of F. This is because in many regards, the derivative of functors between stable categories behave in a far nicer fashion (for instance the derivatives of the identity functor of a stable category are as simple as one could hope, but this is decidedly false in general). Another (related) example of the agreeableness of Goodwillie calculus in the stable setting is the following version of a chain rule (where the map comes from the so called "lax chain rule", which is theorem 2.2.18).

Theorem 1.2.1. *(2.2.33) Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a pair of composable reduced finitary functors between differentiable categories. If $\mathcal{D}$ is stable, there is a natural equivalence*

$$\mu : \partial_*(G) \circ \partial_*(F) \to \partial_*(GF).$$

The above result has some interesting corollaries which reveal an algebraic structure to the chain rule which is seemingly absent from the classical chain rule. The main example of interest to us comes from applying the stable chain rule to the equivalence from proposition 2.3.1, which reveals a crucial link between Goodwillie calculus and Koszul duality (we say nothing more about this subject in the introduction, and direct the reader directly to subsection 2.3 where we introduce Koszul duality, culminating in theorem 2.3.16, which justifies the term duality in our application).

Corollary 1.2.2. *(2.3.12) Let $\mathcal{C}$ and $\mathcal{D}$ be differentiable categories, with $\mathcal{D}$ stable and $K : \mathcal{C} \to \mathcal{D}$ be a reduced finitary functor. Then the derivatives of K are the Koszul dual of the derivatives of $K\Omega^\infty_{\mathcal{C}}$ relative to $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.*

The point of the above result is that for a functor $F : \mathcal{C} \to \mathcal{D}$ to a stable category, it suggests first studying the derivatives of $F \circ \Omega^\infty_{\mathcal{C}}$, and then computing the derivatives of F by Koszul duality. This is promising as, in this case, $F \circ \Omega^\infty$ is a functor between stable categories, and so we suspect it to be quite approachable by the methods of Goodwillie calculus.

In this project we will want to apply these ideas to the notoriously hard to compute Algebraic K-theory. It traces its origins to Grothendieck's proof of the Riemann-Roch theorem, where he introduced K_0, Quillen's definition for general rings of all higher K-groups, Waldhausen definition in terms of a simplicial $S.$ construction and many more which we fail to mention. We will consider K-theory in its modern incarnation provided by Blumberg, Gepner and Tabuada in [4] as a universal object.

Proposition 1.2.3. *(3.2.6) The natural transformation $\Sigma^\infty_+\iota \to K$ of functors $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ exhibits algebraic K-theory as the initial finitary additive invariant under $\Sigma^\infty_+\iota$, the post-composition by $\Sigma^\infty : \mathcal{S} \to \mathrm{Sp}$ of the groupoid core.*

There is, however, an immediate problem in this idea, which is that Goodwillie calculus out of $\mathrm{Cat}^{\mathrm{ex}}$ is trivial, essentially because the suspension functor is 0 in this case by proposition 3.3.1. This is solved by working "around" a stable category $\mathcal{C}$, that is to say viewing K-theory as a functor $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{Sp}$. This creates a new problem that K-theory is no longer reduced, but this is easily fixed by setting $\widehat{K}(\mathcal{D} \to \mathcal{C}) := \mathrm{fib}(K(\mathcal{D}) \to K(\mathcal{C}))$.

By what we discussed above, the natural path to follow is to try and get a handle on the derivatives of $\widehat{K} \circ \Omega^{\infty}_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}}$, which entails understanding the stabilization of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$. This is achieved in the following result, which comes from [16] by Harpaz, Nikolaus and Saunier.

Proposition 1.2.4. *(3.3.9) The stabilization of* $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ *is the category* $\mathrm{BiMod}(\mathcal{C}) \simeq \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$. *Under this identification* Ω^∞ *corresponds to* $\mathrm{Lace}(\mathcal{C}, -) : \mathrm{BiMod}(\mathcal{C}) \to \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$*, which sends a bimodule* $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ *to the category of* M*-laced objects of* $\mathcal{C}$*, i.e. the choice of an object* $x \in \mathcal{C}$ *and a map* $x \to M(x)$.

Our goal is then to understand the derivatives of $\widehat{K} \circ \mathrm{Lace}(\mathcal{C}, -)$, which we rename K^{cyc} for convenience. This actually leads us to one of the most famous theoretical advances in K-theory : the so called "trace methods". This notably includes the Dennis trace map from K-theory to THH, which opened the way to transferring computation from THH to K-theoretic results (which was then refined to the cyclotomic trace which is a map to TC). A textbook account of the classical perspective on this is in the book [9] by Dundas, Goodwillie and McCarthy.
We will follow the considerably more modern treatment from the second half of Ramzi's thesis [30], firmly interpreting the relation between K-theory and THH as one from Goodwillie calculus, culminating in the following result.

Theorem 1.2.5. *(3.4.44) The Dennis trace map* $K \to \mathrm{THH}$ *induces an equivalence of trace theories*

$$\partial_1 K^{\mathrm{cyc}} \simeq \mathrm{THH}\,.$$

The proof is firmly rooted in universal properties, and requires introducing the notion of trace theory, of which both $\partial_1 K^{\mathrm{cyc}}$ and THH are examples. We leave the details of trace theories to subsubsection 3.4.1, and here just point out that the prototypical input of one is the choice of a stable category $\mathcal{C}$ and a cocontinuous functor $M : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$. The above theorem then follows that trace theories are so rigid that if they agree on $(\mathrm{Sp}, \mathrm{Id})$, they must be equivalent. Which up to identifying that the equivalence is the Dennis trace map reduces to the following result.

Corollary 1.2.6. *(3.4.39 and 3.4.42)) There are equivalences*

$$\partial_1 K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{Id}) \simeq \mathbb{S},$$

$$THH(\mathrm{Sp}, \mathrm{Id}) \simeq \mathbb{S}.$$

A recent result which appears in Victor Saunier's thesis [33] states that the first derivative of K-theory, actually determines all higher derivatives.

Theorem 1.2.7. *(3.4.1) There is a Cartesian square*

$$\begin{tikzcd}
P_n K^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] \arrow[d] & P_1 K^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{h}\, C_n} \arrow[d] \\
P_{n-1} K^{\mathrm{cyc}}(\mathcal{C}, M) \arrow[r] & P_1 K^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{t}\, C_n}.
\end{tikzcd}$$

In particular, the nth homogeneous part of K^{cyc} *is given by*

$$THH(\mathcal{C}, M^{\otimes n})_{\mathrm{h}\, C_n}.$$

The next step in understanding the derivatives of $\widehat{K}$ through the lens of Koszul duality as it appears in Goodwillie calculus involves understanding the comonad $\Sigma^{\infty}_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}} \Omega^{\infty}_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}}$. We suspect this comonad to be the one sending a bimodule to the bimodule underlying the associated cofree comonad, whose existence we prove in corollary 4.2.6. We were not fully able to confirm this hunch, but provide the following "conditional" theorem, with a full discussion of the missing pieces in subsection §4.3.

Theorem 1.2.8. *(4.3.5) We have* $\mathrm{coAlg}(\mathrm{coFree}(-))^\omega \simeq \mathrm{Lace}(\mathcal{C}, -)$*. In particular the comonad associated to the right adjoint* $\mathrm{Lace}(\mathcal{C}, -)$ *sends a bimodule* M *to the bimodule underlying the cofree comonad* $\mathrm{coFree}(M)$*.*

Everything which we discussed in this project culminates in the following result, which summarizes how much we are able to undertake of the program Goodwillie calculus suggests to study the derivatives of $\widehat{K}$ (there are some pieces of notation in the following result which did not make an appearance in the introduction, but from the proof of the following statement, it should not be too hard to decipher the statement in detail).

Theorem 1.2.9. *(5.0.1) We have the following equivalence*

$$\partial_*\widehat{K} \simeq \mathrm{coBar}(\phi^{-1}(\mathrm{cr}_*(THH(\mathcal{C}, -^{\otimes *})_{hC_*})), \partial_*(U \circ \mathrm{coFree}_{\mathrm{Bimod}(\mathcal{C})}), \boldsymbol{1}).$$

Which we can rephrase as saying that the derivatives of $\widehat{K}$ *are Koszul dual to the derivatives of* K^{cyc} *with respect to the comodule structure on the cooperad given by the derivatives of the comonad which sends a bimodule to the bimodule underlying the corresponding cofree comonad.*

1.3 Conventions

- When we say category we mean ∞-category, more specifically the quasi-category model as developed by Lurie in [22] and [23]. And similarly, when we call an object a 2-category, we implicitly mean an $(\infty, 2)$-category.

- We denote the category of spaces by $\mathcal{S}$ and of spectra by Sp. And in the latter category we denote the sphere spectrum by $\mathbb{S}$.

- For stable categories there can be ambiguity between whether we mean mapping spaces or mapping spectra. We will write Hom for the former and hom for the latter.

- When we write $[n]$ we mean the set of cardinality n $\{1, ..., n\}$.

- If f is a left adjoint, and we need to refer to its right adjoint without caring to give it a name, we call the right adjoint f^R and similarly in the dual case.

- We often work over $\mathcal{C}$, in which case the mapping spaces are denoted as $\mathrm{Hom}_{/\mathcal{C}}$ with the ambient category left implicit. Furthermore if we are considering maps between $f : \mathcal{D}_0 \to \mathcal{C}$ and $g : \mathcal{D}_1 \to \mathcal{C}$ we will write $\mathrm{Hom}_{\mathcal{C}}(f, g), \mathrm{Hom}_{\mathcal{C}}(\mathcal{D}_0, g), \mathrm{Hom}_{\mathcal{C}}(f, \mathcal{D}_1)$ and $\mathrm{Hom}_{\mathcal{C}}(\mathcal{D}_0, \mathcal{D}_1)$ interchangeably.

- There are certain families of functors which we write uniformly, and let context determine which specific case we are dealing with, these include: forgetful functors which are often denoted by U, projections by π and inclusions which are denoted by ι.

- In the spirit of trying to decolonize mathematical notation, we write the Yoneda embedding for $\mathcal{C}$ with the Japanese hiragana よ$_{\mathcal{C}} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$, potentially omitting the subscript when it is obvious. We also use this notation when we corestrict the Yoneda embedding, for example when considering the canonical map $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

- In a related spirit, we decided to capitalize the term Algebra. We have no pretense to have undertaken a maximal deconstruction of the western eye on mathematics, and the author would be thrilled if any reader cared to point out any other ideas in this initiative.

- Regarding capitalization of coterms, we capitalize only the letters which would be capitalized in the absence of the "co" prefix. I.e. we write coAlgebra, coCartesian fibration, the category of comonads on $\mathcal{C}$ is coMonad$(\mathcal{C})$, etc. We do this on the one hand to respect the cases in which coterms involve a proper noun, on the other for pedagogical clarity as coterms play a crucial role in this project, and we deemed it relevant to make it explicit when one can dualize intuition that the reader might have found in their studies.

2 Goodwillie calculus

2.1 Defining the derivative

Remark 2.1.1. A fair amount of this section was lifted from my master project, which is available on my website [35]. We say this so as to indulge as little as possible in self-plagiarism.

Goodwillie calculus is a theory to approximate complicated functors between ∞-categories by certain simpler functors, in the same spirit as Taylor series approximate sufficiently nice functions $f : \mathbb{R} \to \mathbb{R}$ by polynomials. In fact this analogy is sufficiently tight to support the learner's intuition, and is a valuable source of understanding in Goodwillie calculus.

The first point that needs discussing is to establish which categories are "nice enough" to be covered by our calculus. Indeed, in the same way that classical calculus only speaks of functions between manifolds, and not on any space, Goodwillie calculus deals with a similar restriction. To this end we introduce.

Definition 2.1.2. ([23, 6.1.1.6.]) A category $\mathcal{D}$ is called *differentiable* if:

(i) $\mathcal{D}$ admits all finite limits,

(ii) $\mathcal{D}$ admits sequential colimits, i.e. colimits for $\mathrm{N}(\mathbb{Z})$-shaped diagrams.

(iii) The colimit functor $\mathrm{colim} : \mathrm{Fun}(\mathrm{N}(\mathbb{Z}), \mathcal{D}) \to \mathcal{D}$ is left exact, i.e preserves finite limits.

Example 2.1.3. ([3, 3.1.3.]) A stable presentable category is differentiable. Indeed, presentability is one way to ensure the existence of the (co)limits we require. And the colimit functor which appears in the definition of differentiable commutes with all colimits by virtue of commutativity of colimits with colimits (see [22, 5.5.2.3.] for the ∞-categorical version of this statement). Therefore, it in particular commutes with finite colimits, but in the stable setting this implies that the colimit functor commutes with finite limits as well.

The second point of order when devising such an approximation theory, is to determine what constitutes "nice" functors, i.e. what type of functors will play the role of polynomial functors of degree n. In our case, this role will be filled by so called n-excisive functors. Before defining these, we introduce some vocabulary which will allow us to define n-excisiveness more succinctly.

Definition 2.1.4. ([23, 6.1.1.1.]) Given a (finite) set S, we have $\mathcal{P}(S)$ the 1-*category of subsets of* S corresponding to the poset $(2^S, \subset)$. We can see this as an ∞-category by taking the nerve.
We denote by $\mathcal{P}_{\leq i}(S)$ the full subcategory whose objects are the subset of cardinality less than or equal to i. We define $\mathcal{P}_{>i}(S)$ similarly.

The above definition serves to define n-cubes, which are the fundamental diagram we will need to define n-excisiveness.

Definition 2.1.5. ([23, 6.1.1.2.]) For a category $\mathcal{C}$, an *n-cube* in $\mathcal{C}$ is an $\mathrm{N}(\mathcal{P}(S))$-shaped diagram in $\mathcal{C}$, where S is a finite set of cardinality n. We may also call these S-cubes. We call the functor category $\mathrm{Fun}(\mathrm{N}(\mathcal{P}(S)), \mathcal{C})$ the category of n-cubes in $\mathcal{C}$.

If we decompose S as $T_- \sqcup T \sqcup T_+$, we can obtain a T-cube from an S-cube X by mapping $T_0 \subset T$ to $X(T_- \sqcup T_0)$. The T-cubes obtained this way are called T-faces of X.

Amongst n-cubes, those of relevance are the following.

Definition 2.1.6. ([23, 6.1.1.2.]) An n-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ is called *Cartesian* if the obvious map

$$X(\emptyset) \to \varprojlim_{\emptyset \neq S \subset [n]} X(S)$$

is an equivalence.
We call an n-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ *strongly coCartesian* if it is a left Kan extension of its restriction to $\mathrm{N}(\mathcal{P}([n])_{\leq 1})$.

With this, we can define n-excisiveness.

Definition 2.1.7. ([23, 6.1.1.3.]) A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *n-excisive* if it carries strongly coCartesian $n+1$-cubes to Cartesian $n+1$-cubes. These functors assemble into a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denoted by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.

The legitimacy of these definitions can be assessed in several ways. For our purposes we will content ourselves with the following theorem, saying that there is a well defined "best" n-excisive approximation, which is nicely parallel to the classical theory of Taylor polynomials.

Theorem 2.1.8. *([23, 6.1.1.10]) Let $\mathcal{C}$ be a category with finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. Then the natural inclusion functors $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint which we denote by P_n. Further, these functors are left exact (i.e. preserve finite limits).*

Remark 2.1.9. The proof of the above proceeds by direct construction expressing $P_n F$ as $\mathrm{colim}_{\mathbb{N}} T_n F$, where $T_n F(X)$ is defined as the limit of a diagram built out of evaluations of F at some functor of X. The point of this observation is that it opens the gate to some simple observations of properties of functors which are preserved by n-excisive approximation.

Remark 2.1.10. The asymmetric conditions on the domain and codomain of F can make it somewhat troublesome to consider Goodwillie calculus as something defined on some "category of categories", rather than just on functors between 2 fixed categories. For the time being, this is not troublesome, but in the next section we will briefly need to reconsider this issue.

The n-excisive approximations for varying n are related by the following result.

Corollary 2.1.11. *([23, 6.1.1.4.]) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories. Assume that $\mathcal{C}$ admits finite colimits and finite limits. If F is m-excisive then it is n-excisive for all $n \geq m$.*

This result implies that there is a so-called *Taylor tower* of approximations

$$\begin{tikzcd}
 & P_\infty F \simeq \lim P_n F \arrow[d] \\
 & \vdots \arrow[d] \\
 & P_1 F \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uuur] & P_0 F
\end{tikzcd}$$

Wishing to hang on to the analogy with classical calculus, one can reasonably be confused by the fact that we have an analog for Taylor series, but have not yet defined the derivative. In fact, we will spend the rest of this subsection sketching the path to a definition of derivatives. Towards this goal, we will introduce some vocabulary to talk about the fiber of the maps between successive layers of the tower, which we record in the following definition.

Definition 2.1.12. We denote by $D_n F$ the fiber of the canonical map $P_n F \to P_{n-1} F$ and call it the *nth homogeneous part of F*.

Pursuing the analogy with classical calculus, and as suggested by the nomenclature, we would like to somehow say that $D_n F$ is n-homogeneous. The appropriate definition of this notion in our context is included in the following.

Definition 2.1.13. ([23, 6.1.2.1.]) Let $\mathcal{C}$ be a category that admits finite colimits and has a final object, $\mathcal{D}$ be a differentiable category and $F : \mathcal{C} \to \mathcal{D}$ a functor. We call F *n-reduced* if $P_{n-1} F$ sends any object of $\mathcal{C}$ to a final object of $\mathcal{D}$. And we call F *n-homogeneous* if it is n-reduced and n-excisive. We assemble the 1-reduced (or often just *reduced*) functors into a full subcategory of the functor category denoted by $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ and similarly for the n-homogeneous functors, which we denote by $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

The fact that D_nF is in fact n-homogeneous follows from the fact that the various P_n preserve fiber sequences, by virtue of being left exact. The functor D_nF corresponds to $\frac{f^{(n)}(0)}{n!}x^n$ in classical calculus, since taking the fiber can be thought of a subtraction, and in particular D_nF does not quite correspond to the nth derivative of F. Naively, we might hope to recover $f^{(n)}(0)x^n$ by somehow generalizing "multiplication by $n!$", but there is no obvious and satisfactory way to do this. Hence, we get rid of the denominator differently, by taking the "nth cross-effect".

Remark 2.1.14. The nth cross-effect is something one can consider in classical calculus, but to the author's knowledge does not belong to the standard curriculum, and as it is quite an enjoyable motivation, we include a small word about it here. The nth cross-effect of $f(x)$ replaces x by the sum of n independent variables $\{x_i\}_{i=1}^n$, and then extracts the coefficient in front of $\prod_{i=1}^n x_i$ by the following formula

$$\mathrm{cr}_n(f)(x_1, ..., x_n) = (-1)^n \sum_{S \subset [n]} (-1)^{|S|} f(\sum_{i \in S} x_i).$$

So for example, applied to $\frac{f^{(2)}(0)}{2}x^2$, we get

$$\frac{f^{(2)}(0)}{2}(x_1 + x_2)^2 - \frac{f^{(2)}(0)}{2}x_1^2 - \frac{f^{(2)}(0)}{2}x_2^2 = f^{(2)}(0)x_1x_2.$$

And this suitably recovers the 2nd derivative of f.

Following this motivation, we strive to define ∂_nF as a a functor of n-variables, which is "multilinear", which requires defining this notion. This requires a multivariable version of P_n and the appropriate definitions so that we can even ask for such a thing to exist. First, we need a notion of $\vec{n}$-excisive, for a multi-integer $\vec{n} \in \mathbb{N}^m$.

Definition 2.1.15. ([23, 6.1.3.1.]) Suppose we are given ∞-categories $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ which admit pushouts and $\mathcal{D}$ a category with finite limits. Given a sequence of integers $\vec{n} = (n_1, ..., n_m), n_i \in \mathbb{N}$, we call a functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ *$\vec{n}$-excisive* if the composition

$$\mathcal{C}_j \to \mathcal{C}_j \times \prod_{i \neq j, i=1}^{m} \{X_j\} \to \prod_{i=1}^{m} \mathcal{C}_i \xrightarrow{F} \mathcal{D}$$

is n_j-excisive for any choice of j and objects $\{X_i\}_{i \neq j}$ with $X_i \in \mathcal{C}_i$. In simpler terms, we call F $\vec{n}$-excisive if it is n_i-excisive in its ith variable.
We denote by $\mathrm{Exc}^{\vec{n}}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$ the collection of all $\vec{n}$-excisive functors.

Remark 2.1.16. Notice that in the notation $\mathrm{Exc}^{\vec{n}}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$, the fact that we write $\prod_{i=1}^m \mathcal{C}_i$ as such, rather than writing $\mathcal{C} = \prod_{i=1}^m \mathcal{C}_i$, is crucial. This mild imperfection of our choice of symbolic representation will always be present when we talk about multivariable categories of functors, but we will not point out this subtlety each time.

Unsurprisingly, we have the following theorem which opens the door to multivariable Goodwillie calculus, which we require to speak about derivatives in the single variable context.

Proposition 2.1.17. *([23, 6.1.3.6.]) Let $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ be categories which admit finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. Then, for any sequence of integers $\vec{n}$, the inclusion $\mathrm{Exc}^{\vec{n}}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint $P_{\vec{n}}$, which is left exact.*

The above theorem then gives obvious definitions of $\vec{n}$-reduced and $\vec{n}$-homogeneous, the most important case of which we record in the following definition.

Definition 2.1.18. ([23, 6.1.3.7.]) Let $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ be categories which admit finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. We call a functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{C}$ *multilinear* if it is $(1, ..., 1)$-homogeneous. We write $\mathrm{Fun}^{mlin}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$ for the category of multilinear functors.

As we now have categories of reduced and homogeneous functors, which admit inclusions into the category of all functors, we might wonder about potential adjoints. In particular, we might wish for a reduction functor $\mathrm{Fun}(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}_*(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D})$. We can further motivate this by referring back to the classical cross effect, which distinguishes itself from $f(x_1 + ... + x_n)$ by the property that if any one of the x_i is 0, then $\mathrm{cr}_n(f)(x_1, ..., x_n) = 0$. So that in time, for a functor F, we might hope to define its nth cross effect $\mathrm{cr}_n(F) : \mathcal{C}^{\times n} \to \mathcal{D}$ as the reduction of the functor that sends $\{X_i\}_{i=1}^{n}$ to $F(X_1 \oplus \cdots \oplus X_n)$.
In fact, we do have a right adjoint to the inclusion of reduced functors, by the following result.

Proposition 2.1.19. *([23, 6.1.3.18.]) Let $\mathcal{C}_1, ..., \mathcal{C}_m$ be categories which admit a final object and let $\mathcal{D}$ be a pointed category with finite limits. Then the inclusion*

$$\mathrm{Fun}_*(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D})$$

admits a right adjoint denoted by $\mathrm{Red} : \mathrm{Fun}(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}_*(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D})$.

Before introducing the cross effect, we introduce one more piece of notation, as we will want to view the cross-effect as a symmetric functor, and we need to know what these are.

Definition 2.1.20. ([23, 6.1.4.1.]) We denote the construction which encodes unordered n-tuples by $K^{(n)}$. Explicitly, this is given as the (homotopy) orbits of the natural action of S_n on $K^{\times n}$.
This allows us to define a *symmetric n-ary functor* as a functor $\mathcal{C}^{(n)} \to \mathcal{D}$. These obviously assemble into a category $\mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. A symmetric n-ary functor always has an underlying functor $\mathcal{C}^n \to \mathcal{D}$, and so we define a symmetric n-ary functor to be reduced if it is underlying functor is reduced (in each variable). The reduced symmetric n-ary functors assemble into a category $\mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ which is a full subcategory of $\mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$.

We are now finally ready to define the n-th cross effect.

Definition 2.1.21. ([23, 6.1.4.5.]) Consider a category with finite coproducts and final object $\mathcal{C}$, and denote by $q : \mathcal{C}^{(n)} \to \mathcal{C}$ the functor which sends an n-tuple to their coproduct. Given a functor $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is pointed with finite limits, we denote by cr_n the reduction of $F \circ q : \mathcal{C}^{(n)} \to \mathcal{D}$.

Remark 2.1.22. We are technically being somewhat sloppy when it comes to the distinction between $\mathcal{C}^n$ and $\mathcal{C}^{(n)}$, for example reduction is a priori only defined for functors out of the former, not the latter. As this section serves to recall theory best learned elsewhere, we indulge in this abuse so as to not lose the trees amongst the forest.

What makes the nth cross-effect relevant to the study of homogeneous functors (such as $D_n F$ recalling that our through line is defining the derivative) is the following result.

Theorem 2.1.23. *([23, 6.1.4.7.]) Let $\mathcal{C}$ be a pointed category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category. Then we have a fully faithful embedding*

$$\mathrm{cr}_{(n)} : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}).$$

The essential image of $\mathrm{cr}_{(n)}$ is the full subcategory $\mathrm{SymFun}^{n,mlin}(\mathcal{C}, \mathcal{D})$ of those functors $E : \mathcal{C}^{(n)} \to \mathcal{D}$ whose underlying functor $E : \mathcal{C}^n \to \mathcal{D}$ is multilinear.

We need one more result before defining the derivative.

Theorem 2.1.24. *([23, 6.2.3.21.] and [23, 6.2.3.22.]) Let $\{\mathcal{C}_i\}_{i \in I}$ be a finite collection of pointed differentiable categories and let $\mathcal{D}$ be a differentiable category. Then the construction $f \mapsto \Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{i \in I} \Sigma^\infty_{\mathcal{C}_i}$ defines an equivalence $\phi : \mathrm{Fun}^{mlin}_{*,\omega}(\prod_{i \in I} \mathrm{Sp}(\mathcal{C}_i), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Fun}^{mlin}_{*,\omega}(\prod_{i \in I} \mathcal{C}_i, \mathcal{D})$ (the subscripts $*$ and ω respectively mean reduced and finitary in each variable).*

Remark 2.1.25. We point out that the above theorem fits in a more general principle of Goodwillie calculus, that between stable categories Goodwillie calculus is generally simpler than between arbitrary differentiable categories. In fact, there are reasons to even consider the stable case to subsume the general one. This can be thought as analog to the fact that analysis on manifolds can be understood from analysis on $\mathbb{R}^n$. In so far as the connection between calculus on manifolds and on Euclidean space is captured by the tangent bundle, the analogous idea in Goodwillie calculus is made precise in [2].

And so we finally define the nth derivative of a functor.

Definition 2.1.26. Let $F : \mathcal{C} \to \mathcal{D}$ be a reduced finitary functor between complete and cocomplete differentiable categories (these hypothesis could in theory be relaxed). By the above discussion, the nth cross-effect of D_nF fits in the following commutative diagram, where we define the top arrow to be the *nthe derivative of* F, denoted by $\partial_n F$

$$\begin{tikzcd} \mathrm{Sp}(\mathcal{C})^n \arrow[r] & \mathrm{Sp}(\mathcal{D}) \arrow[d, "\Omega^\infty_{\mathcal{D}}"] \\ \mathcal{C}^n \arrow[u, "(\Sigma^\infty_{\mathcal{C}})^n"] \arrow[r, "\mathrm{cr}_n(D_nF)"] & \mathcal{D} \end{tikzcd} .$$

Remark 2.1.27. The functor $\partial_n F$ is multilinear and reduced, so that it preserves final objects and fiber sequences in each coordinate, i.e it is exact. Since we are working stably, this implies that $\partial_n F$ preserves finite colimits in each variable. Furthermore, this functor is finitary in each variable, which all together imply that $\partial_n F$ is cocontinuous in each variable.
This in particular means that in the case that $\mathrm{Sp}(\mathcal{C})$ is generated under colimits by a single object S, the nth derivative of F is entirely determined by the object with S_n-action $\partial_n F(S, ..., S)$.

2.2 Chain Rule and consequences

In this section, we give a tour of some core results of Blans and Blom in their paper [3] on the chain rule in Goodwillie calculus. The chain rule is a highly desirable feature in any emulation of classical calculus, at the basest level this is because it grants access to computations of object appropriately built out of simpler ones. In the categorical context of Goodwillie calculus, the chain rule says something "more" than classically, in that it becomes an "algebraically charged" statement.
For now, we justify this claim by the inclusion in passing of the following statement.

Proposition 2.2.1. *([3, 4.4.9.]) Let $\mathcal{C}$ be a pointed presentable differentiable category, and write* $\mathrm{BiMod}_{\partial_* \mathrm{Id}_{\mathcal{C}}, \partial_* \mathrm{Id}_{\mathcal{C}}}$ *for the category of $\partial_* \mathrm{Id}_{\mathcal{C}}$ bimodules in* $\mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{C}))$. *Then the Goodwillie derivatives refine to a strong monoidal functor*

$$\mathrm{Fun}_{*,\omega}(\mathcal{C}, \mathcal{C}) \to \mathrm{BiMod}_{\partial_* \mathrm{Id}_{\mathcal{C}}, \partial_* \mathrm{Id}_{\mathcal{C}}} .$$

In particular, it sends monads to operads and comonads to cooperads.

For our purposes, the algebra which the chain rule reveals will not be the above statement, but rather a Koszul duality statement. In fact, within the context of this document, this section can be seen as building up to the Koszul duality statement. Therefore, we ask the reader to be patient, and wait till the end of this section for the second justification of our claim that the chain rule contains within it a fair amount of algebra.

Remark 2.2.2. It is perhaps somewhat frustrating that this algebraic dimension to the chain rule is not classically apparent. This is essentially because 0-categorically there are no "arrows" which could encode multiplication. As a consolation prize for the absence of algebras in monoidal 0-categories (monoids), we can notice that we can still speak of idempotent algebras (idempotent elements). In the case of the monoid of smooth functions $\mathbb{R}^n \to \mathbb{R}^n$, considering an idempotent $f : \mathbb{R}^n \to \mathbb{R}^n$, we see some faint hint of the algebraic nature of the chain rule in that it gives the following relation

$$df_{f(x)} df_x = df_x.$$

2.2.1 Some set-up

The final form of the chain rule will say that we can see ∂_* as a strong 2-functor, from an appropriate category of (stable) categories to a category whose morphisms will be symmetric sequences of some kind[1]. In particular, to land in a category of symmetric sequence, the final form of the chain rule requires that we consider all derivatives at once (i.e. we are not considering the ∂_n individually) and requires we "globalize" the various constructions of the previous section, so that we can consider more than a single functor category at a time.

Remark 2.2.3. The latter should not be two surprising, because at the minimum, the chain rule for $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ requires that we can handle $\mathrm{Fun}_{*,\omega}(\mathcal{C}, \mathcal{D}), \mathrm{Fun}_{*,\omega}(\mathcal{D}, \mathcal{E})$ and $\mathrm{Fun}_{*,\omega}(\mathcal{C}, \mathcal{E})$ simultaneously.

Remark 2.2.4. From here on out, we consider only presentable, pointed differentiable categories, and do not recall this assumption throughout. We also point out in passing that with the first two assumptions, the differentiable hypothesis can be rephrased as asking that for any filtered category I, the functor colim_I is left exact (preserves finite limits).

We first discus the point of wanting to treat all derivatives at once, which will in particular lead us to consider $\mathrm{SymFun}(\mathcal{C}, \mathcal{D}) := \coprod_{n \geq 0} \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. It is not hard to see that $\mathrm{SymFun}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathrm{Sym}^\times(\mathcal{C}), \mathcal{D})$ where $\mathrm{Sym}^\times(\mathcal{C}) = \coprod_{n \geq 0} \mathcal{C}^{(n)}$.

Remark 2.2.5. The construction sending $\mathcal{C}$ to $\mathrm{Sym}^\times(\mathcal{C})$ is in fact the free symmetric monoidal category on $\mathcal{C}$, in particular, for a functor $F : \mathcal{C} \to \mathcal{D}$, we get a functor $\mathrm{Sym}^\times(F) : \mathrm{Sym}^\times(\mathcal{C}) \to \mathrm{Sym}^\times(\mathcal{D})$.

In light of the specific properties of $\partial_* F$, we introduce the following definitions, giving us some relevant vocabulary.

Definition 2.2.6. ([3, 3.1.23.]) Let $\mathcal{C}, \mathcal{D}$ be presentable categories. We refer to the restriction of a functor $F : \mathrm{Sym}^\times(\mathcal{C}) \to \mathcal{D}$ to $\mathcal{C}^{(n)}$ as the nth component of F and denote it by F_n. We will often say a functor F has some property P if it each component has this property.
At times it will be relevant to model $\mathrm{Sym}^\times(\mathcal{C})$ as the colimit over the groupoid core of finite sets of the functor which sends I to $\mathcal{C}^I$. The equivalence with the previous formula comes from the inclusion of a skeleton always being cofinal. For a functor $F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$ this gives an obvious way to speak of F_I, the Ith component of F.
We call a functor $F : \mathrm{Sym}^\times(\mathcal{C}) \to \mathcal{D}$ a *functor symmetric sequence* (or just symmetric sequence at times), if the components F_n preserve colimits separately in each variable. We denote the full subcategory of $\mathrm{SymFun}(\mathcal{C}, \mathcal{D})$ spanned these by $\mathrm{SymFun}^L(\mathcal{C}, \mathcal{D})$. If $\mathcal{C}$ and $\mathcal{D}$ are furthermore pointed, we add the subscript ≥ 1 to denote those symmetric sequences whose 0th component is constant equal to the 0 object of $\mathcal{D}$ and we call these *positive.*

Remark 2.2.7. When working with symmetric functor sequences between presentable categories, by universal property of the tensor product on Pr^L we can also describe $\mathrm{SymFun}^L(\mathcal{C}, \mathcal{D})$ as $\mathrm{Fun}^L(\mathrm{Sym}(\mathcal{C}), \mathcal{D})$, where $\mathrm{Sym}(\mathcal{C})$ is the free commutative algebra in Pr^L, with an explicit formula given by $\bigoplus_{n \geq 0} \mathcal{C}^{\otimes n}_{h\Sigma_n}$.

It is clear for the reduction and the direct sum, and thus for the the cross effect, taking the coproduct over n yields a version of these constructions which applies to $\mathrm{SymFun}(\mathcal{C}, \mathcal{D})$. For example, the inclusion $\mathrm{SymFun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}(\mathcal{C}, \mathcal{D})$ admits a right adjoint red, which is a coproduct over n of the reduction for functors $\mathcal{C}^{(n)} \to \mathcal{D}$. For the cross effect, this idea yields a functor $\mathrm{cr} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}_*(\mathcal{C}, \mathcal{D})$.

We judiciously did not include P_n in our brief discussion of taking constructions from the previous section "all n at at time". We could simply take the coproduct of the P_n yielding a functor

$$P_* : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \coprod_{n \geq 0} \mathrm{Exc}^n(\mathcal{C}, \mathcal{D}).$$

[1] In case one wants to consider a chain rule not just for stable categories, one needs to replace symmetric sequences by suitable bimodules. We will not say more about the general chain rule, so we will not need to develop this theory.

We could define D_* in a similar direct fashion. Already here, we might be struck with a hope for a different perspective, as defining D_* from P_* is somewhat clunky. And therefore, we might consider this a disappointing way to work "all at n once", since the simple connection between D_n and P_n becomes muddled when considering D_* and P_*.

Luckily, we do have another path to define the nth derivative courtesy of the following result

Proposition 2.2.8. *([23, 6.1.3.23.]) Let $\mathcal{C}$ be an ∞-category which admits colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. Then we have*

$$P_{1,\ldots,1}\,\mathrm{cr}_n(F) \simeq \mathrm{cr}_n(P_n(F)),$$

which implies

$$\mathrm{cr}_n(D_n(F)) \simeq P_{1,\ldots,1}(\mathrm{cr}_n(F)).$$

Which using the next light repetition of theorem 2.1.24, gives access to ∂_* as the composition of $\mathrm{cr} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}_*(\mathcal{C}, \mathcal{D})$ with component-wise multilinearization $P_{\vec{1}} : \mathrm{SymFun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}_*^{mlin}(\mathcal{C}, \mathcal{D})$.

Proposition 2.2.9. *([3, 3.1.26.]) Let $\mathcal{C}$ and $\mathcal{D}$ be presentable categories. The functor*

$$\mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) \to \mathrm{SymFun}_\omega^{mlin}(\mathcal{C}, \mathcal{D})$$

given by sending F to $\Omega^\infty_{\mathcal{D}} \circ F \circ \mathrm{Sym}^\times(\Sigma^\infty_{\mathcal{C}})$ is an equivalence of categories.

What all of this has achieved, is that we have a model for the derivative "for all n at once", as the composition of two functors which are suitably "all n at once" (where we are indulging in a minor abuse in assuming that $P_{\vec{1}}$ lands in $\mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$ using the above theorem).
The reason we went through this effort is so that defining ∂_* as a functor of 2-categories, and then studying it's basic properties (which in time will be the chain rule) becomes more approachable. The remaining set-up for this venture is to say a word about what we precisely want the domain and codomain of ∂_* to be. Regarding the domain, As was foreshadowed to in remark 2.1.10, if we wish to conduct Goodwillie calculus on a category of categories, we need to be weary that every category in consideration satisfies every condition needed to apply the results of the previous section. We accomplish this with the following definition.

Definition 2.2.10. ([3, 3.3.1.]) We denote by Diff the locally full 2-subcategory of Cat_∞ on pointed, presentable differentiable categories and reduced finitary functors between them.

Remark 2.2.11. To fully understand the chain rule requires some understanding of $(\infty, 2)$-categories. We will not undergo an in depth introduction or recollection of this subject here, hoping to justify this decision by the belief that carrying over intuition from classical 2-category theory suffices to read the statements without finding oneself overwhelmed by terminology.
For the reader who wishes for a more precise handle on the required $(\infty, 2)$-category theory, we recommend the appendix [3, A], which is an essentially self-contained introduction to the subject, specifically geared towards proving the chain rule. We also recommend the appendix [20, D], which takes the arguably more intuitive perspective that 2-categories are just categories enriched in 1-categories, but covers material geared towards different applications in mind.
We will nonetheless have to use some 2-categorical ideas and some specific theorems to obtain the functors we wish to study. Indeed, just as 1-categorically, when considering $(\infty, 1)$-categories, writing down all coherence is too tedious, and so we need theorems or prior constructions to define functors, obtaining functors of $(\infty, 2)$-categories will require theorems, rather than explicit definitions. Rather than make a separate section for these results, we will include them as they come up.

For the codomain, we somehow need a 2-category which captures functor symmetric sequences between categories, i.e. the category between 0-cells $\mathcal{C}$ and $\mathcal{D}$ should be $\mathrm{SymFun}^L(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^L(\mathrm{Sym}(\mathcal{C}), \mathcal{D})$. The problem is the asymmetry between $\mathrm{Sym}(\mathcal{C})$ and $\mathcal{D}$, which makes defining composition unclear.
We are saved by the following result.

Proposition 2.2.12. *([3, 3.2.2.]) Let $\mathcal{C}$ be a presentable category. Then* $\mathrm{Sym}(\mathcal{C})$ *is the underlying category of the cofree cocommutative coAlgebra on $\mathcal{C}$ in the 2-category Pr^L.*

Proof. The category of cocommutative coAlgebras of Pr^L is the opposite of the category of algebras of Pr^R, since for any 2-category $\mathbf{C}$, we have $cocAlg(\mathbf{C}) := cAlg(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$ and we have an anti-equivalence $Pr^L \simeq (Pr^R)^{\mathrm{op}}$ which can be found as [22, 5.5.3.4.]. Since the statement we seek to prove is an objectwise one, the discrepancy in the direction of arrows is irrelevant to us, so that it suffices to prove that the explicit formula $\bigsqcup_{n\geq 0} \mathcal{C}^{\otimes n}_{h\Sigma_n}$ for $\mathrm{Sym}(\mathcal{C})$ is the same as the explicit formula for the free commutative Algebra of Pr^R. By [3, 2.2.15.] it suffices that we know that Pr^R admits all colimits indexed by small spaces and the tensor product of presentable categories preserves these in each variable separately. This follows, since colimits in Pr^R are computed as limits in Pr^L, and when the diagram is a small space, these are in turn colimits in Pr^L, which are preserved in each variable by the tensor product of presentable categories. □

Remark 2.2.13. The property that limits and colimits indexed by small spaces in Pr^L are the same is called the ambidexterity of Pr^L. This can be found as [3, 2.3.6.], which is very much related to the anti-equivalence of Pr^L and Pr^R.
We mention in passing that in the course of the above proof, we use only the 1-categorical anti-equivalence $Pr^L \simeq (Pr^R)^{\mathrm{op}}$. But this result can be strengthened to the 2-categorical statement $Pr^L \simeq (Pr^R)^{\mathrm{co,op}}$, which can be found as [20, D.3.17.].

What the above result achieves is the equivalence

$$\mathrm{SymFun}^L(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^L(\mathrm{Sym}(\mathcal{C}),\mathcal{D}) \simeq \mathrm{Fun}^L_{coAlg}(\mathrm{Sym}(\mathcal{C}),\mathrm{Sym}(\mathcal{D})).$$

This motivates the following definition.

Definition 2.2.14. ([3, 3.2.3.]) Denote by Pr^{Sym} to be the full subcategory of the 2-category $cocAlg(Pr^L)$ spanned by the cofree cocommutative coAlgebras $\mathrm{Sym}(\mathcal{C})$. We will still think of the object $\mathrm{Sym}(\mathcal{C})$ as $\mathcal{C}$. There is an obvious definition of $\mathbf{C}^{\mathrm{Sym}}$ for any full subcategory $\mathbf{C}$ of Pr^L.
We use the subscript ≥ 1 when we consider only functors $\mathrm{Sym}(\mathcal{C}) \to \mathrm{Sym}(\mathcal{D})$ whose 0th component is constant equal to the 0 object of $\mathrm{Sym}(\mathcal{D})$ (when this makes sense).

The reason why (a subcategory of) Pr^{Sym} is a good candidate for the codomain of the derivative functor, if we want functoriality to capture the chain rule, is by the following understanding of composition in Pr^{Sym}, which says that the composition defined by the above discussion is the so-called "composition product" of symmetric sequences.

Proposition 2.2.15. *([3, 3.2.10.]) Let $F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$ and $G : \mathrm{Sym}(\mathcal{D}) \to \mathcal{E}$ be positive functor symmetric sequences between presentable pointed categories, and denote by $G \circ F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{E}$ the symmetric functor sequence which comes from composition in* $\mathrm{Pr}^{\mathrm{Sym}}$*. Then we have the following formula*

$$(G \circ F)_I \simeq \bigsqcup_{E\in \mathrm{Part}(I)} G_E \circ \{F_J\}_{J\in E}.$$

Where $\mathrm{Part}(I)$ *is the set of partitions of the set I.*

Remark 2.2.16. It is perhaps not obvious why taking G to be the derivatives of some g, and F to be the derivatives of some f, why the above formula should be the derivatives of $g \circ f$. This can be motivated by the following direct consequence of iterating the classical chain rule

$$(g \circ f)^{(n)}(x) = \sum_{\pi\in \mathrm{Part}([n])} f^{(|\pi|)}(g(x)) \prod_{B\in\pi} g^{(|B|)}(x).$$

2.2.2 The lax chain rule

One might hope that the next step in defining the derivative as a 2-functor is to globalize cr and $P_{\vec{1}}$ into functors of 2-categories, and then we define the derivative as the composite $P_{\vec{1}} \circ \mathrm{cr}$. We however encounter one more problem, which is that the cross-effect's codomain and thus multilinearization's domain is of the form $\mathrm{Fun}(\mathrm{Sym}(\mathcal{C}),\mathcal{D})$, and this cannot be obviously made into the 1-category of morphisms between $\mathcal{C}$ and $\mathcal{D}$ in some 2-category. One might hope this problem is the same as the one for the desired codomain of ∂_*, which we fixed by proposition 2.2.12. But this proposition crucially used the restriction to cocontinuous functors, so that we could use $\mathrm{Fun}^L(\mathrm{Sym}(\mathcal{C}),\mathcal{D}) \simeq \mathrm{Fun}^L(\mathrm{Sym}(\mathcal{C}),\mathrm{Sym}(\mathcal{D}))$, which enabled definition 2.2.14.

We therefore need one more idea before being able to define the derivative as a functor of 2-categories. We do this by formally adjoining finite colimits sending $\mathcal{C}$ to $\mathcal{P}^{\mathrm{fin}}(\mathcal{C})$, sending a functor $\mathrm{Sym}(\mathcal{C}) \to \mathcal{D}$ to a functor $\mathrm{Sym}(\mathcal{P}^{\mathrm{fin}}(\mathcal{C})) \to \mathcal{P}^{\mathrm{fin}}(\mathcal{D})$. If we are considering categories admitting filtered colimits, and the original functor was reduced and finitary, then the new functor is cocontinuous. So that we can once more use proposition 2.2.12 and thus have a codomain which arises as a 1-category of morphisms in a 2-category. In summary, we consider the cross effect as a functor $\mathrm{cr} : \mathrm{Fun}(\mathcal{C},\mathcal{D})_{\omega,*} \to \mathrm{SymFun}_{*,\omega,\geq 1}(\mathcal{C},\mathcal{D})$, and in order for the codomain to have a chance at being globalizable we post-compose this with the embedding $\iota : \mathrm{SymFun}_{*,\omega,\geq 1}(\mathcal{C},\mathcal{D}) \to \mathrm{SymFun}^L_{\geq 1}(\mathcal{P}^{\mathrm{fin}}(\mathcal{C}),\mathcal{P}^{\mathrm{fin}}(\mathcal{D}))$. We record this in the following definition.

Definition 2.2.17. ([3, 3.3.6.]) We call the category $\mathrm{SymFun}^L_{\geq 1}(\mathcal{P}^{\mathrm{fin}}(\mathcal{C}),\mathcal{P}^{\mathrm{fin}}(\mathcal{D}))$ the category of *formal (functor) symmetric sequences* from $\mathcal{C}$ to $\mathcal{D}$.

We denote cross-effect post-composed by the embedding ι by $\widetilde{\mathrm{cr}}$ and call it the formal cross-effect. For this strategy to work we will also need a "formal multilinearization" $\widetilde{P_{\vec{1}}} : \mathrm{SymFun}^L_{\geq 1}(\mathcal{P}^{\mathrm{fin}}(\mathcal{C}),\mathcal{P}^{\mathrm{fin}}(\mathcal{D})) \to \mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}),\mathrm{Sp}(\mathcal{D}))$, which will be achieved by precomposing $P_{\vec{1}}$ by the left adjoint of ι. Throughout all of this, one needs to pay attention that we are defining the formal cross-effect and formal multilinearization in such a way that $\widetilde{P_{\vec{1}}} \circ \widetilde{\mathrm{cr}} \simeq P_{\vec{1}} \circ \mathrm{cr}$. From a study of the "2-categorical niceness" of $\widetilde{\mathrm{cr}}$ and of $\widetilde{P_{\vec{1}}}$, one derives the following result, yielding the comparison map which will eventually lead to the stable chain rule.

Theorem 2.2.18. *There exists a lax 2-functor*

$$\partial_* : \mathrm{Diff} \to Pr^{\mathrm{Sym}}_{St,\geq 1}$$

which sends a differentiable category to its stabilization $\mathrm{Sp}(\mathcal{C})$ *and on mapping categories is given by the Goodwillie derivatives functor.*

Our first undertaking for the strategy sketched above is to say a small further word about ι and its left adjoint. For this, let $\mathcal{D}$ be a pointed presentable differentiable category. The functor $よ : \mathcal{D} \to \mathcal{P}^{\mathrm{fin}}(\mathcal{D})$ preserves limits and filtered colimits, and thus has a left adjoint h. In particular $\mathcal{D}$ is a reflective localization of $\mathcal{P}^{\mathrm{fin}}(\mathcal{D})$. Post composition with this adjunction yields an adjunction between $\mathrm{SymFun}_{*,\omega,\geq 1}(\mathcal{C},\mathcal{D})$ and $\mathrm{SymFun}^L_{\geq 1}(\mathcal{C},\mathcal{P}^{\mathrm{fin}}(\mathcal{D}))$, which by universal property of $\mathcal{P}^{\mathrm{fin}}(\mathcal{D})$ (see [3, 3.3.7.]) is equivalent to $\mathrm{SymFun}^L_{\geq 1}(\mathcal{P}^{fin}(\mathcal{C}),\mathcal{P}^{\mathrm{fin}}(\mathcal{D}))$. In summary, we have an adjunction

$$L : \mathrm{SymFun}^L_{\geq 1}(\mathcal{P}^{fin}(\mathcal{C}),\mathcal{P}^{\mathrm{fin}}(\mathcal{D})) \leftrightarrows \mathrm{SymFun}_{*,\omega,\geq 1}(\mathcal{C},\mathcal{D}) : \iota.$$

Before globalizing $\widetilde{\mathrm{cr}}$ and $\widetilde{P_{\vec{1}}}$, we need the 2-category which will be the latter's domain and the former's codomain. It turns out we will in fact only need a 2-precategory[2]. To obtain this we use the following result.

Proposition 2.2.19. *([3, A.3.19.]) Let S be a space, $\boldsymbol{X}$ be a 2-precategory, whose space of objects is $\boldsymbol{X}_0$ and $\phi : S \to \boldsymbol{X}_0$ be a map of spaces. Then there is a 2-precategory whose space of objects is S and whose mapping categories between s_1 and s_2 is $\boldsymbol{X}(\phi(s_1),\phi(s_2))$.*

Using this on the 2-category of formal symmetric sequences and the space of differentiable categories will give the 2-precategory we need.

[2]The reader unfamiliar with this terminology loses sufficiently little in paying no mind to the prefix added "pre" in 2-precategory. The only point of note is that every 2-category is a 2-precategory, but not the other way around.

Definition 2.2.20. Let Pr_0 be the space of presentable categories, i.e. the space of objects of Pr^L and of Pr^{Sym}. Let Diff_0 be the space of objects of Diff. Consider the morphism $\alpha : \text{Diff}_0 \to Pr_0$ which sends $\mathcal{C}$ to $\mathcal{P}^{\text{fin}}(\mathcal{C})$. Then we define $\text{Diff}^{\text{F Sym}}$ to be $\alpha^* Pr^{\text{Sym}}$. We add a subscript ≥ 1 when we consider the 2-precategory whose mapping categories are spanned by the formal symmetric sequences.

We are now ready to state the result which globalizes the first of our two functors.

Proposition 2.2.21. *([3, 3.3.13.]) There exists a lax 2-functor* $\widetilde{\text{cr}} : \text{Diff} \to \text{Diff}^{\text{F Sym}}_{\geq 1}$*, that is the identity on objects and is given on mapping categories as*

$$\widetilde{\text{cr}} : \text{Fun}_{*,\omega}(\mathcal{C}, \mathcal{D}) \xrightarrow{\text{cr}} \text{SymFun}_{*,\omega,\geq 1}(\mathcal{C}, \mathcal{D}) \xrightarrow{\iota} \text{SymFun}^L_{\geq 1}(\mathcal{P}^{fin}(\mathcal{C}), \mathcal{P}^{\text{fin}}(\mathcal{D})).$$

We only provide a sketch of the proof. The 2-categorical tool which is at the core of the proof is the following result.

Proposition 2.2.22. *([3, A.3.16.]) Let* $F : \boldsymbol{X} \to \boldsymbol{Y}$ *be an oplax functor, which is an equivalence on objects, and the induced components on mapping categories all have right adjoints* $F_{x_1,x_2} : \boldsymbol{X}(x_1, x_2) \to \boldsymbol{Y}_{y_1,y_2}$*. Then there exists a lax functor* $R : \boldsymbol{Y} \to \boldsymbol{X}$ *which on objects is inverse to* F *and on mapping categories is right adjoint to* F_{x_1,x_2}*.*

This result seems like a good path to follow as $\widetilde{\text{cr}} : \text{Fun}_{*,\omega}(\mathcal{C}, \mathcal{D}) \to \text{SymFun}^L_{\geq 1}(\mathcal{P}^{fin}(\mathcal{C}), \mathcal{P}^{\text{fin}}(\mathcal{D})$ is right adjoint to $(h \circ - \circ \text{よ}) \circ q_!$, where $q : \text{SymFun}^L_{\geq 1}(\mathcal{P}^{\text{fin}}(\mathcal{C}), \mathcal{P}^{\text{fin}}(\mathcal{D})) \to \text{Fun}_{*,\omega}(\mathcal{P}^{\text{fin}}(\mathcal{C}), \mathcal{P}^{\text{fin}}(\mathcal{D}))$ is the component wise direct sum functor. This should not be too surprising since the ordinary cross-effect cr is also a right adjoint and passing to the formal situation only adds finite colimits, which by the intuition provided by the adjoint functor theorem, should not have any impact on being a right adjoint. The precise identification of the right adjoint of $\widetilde{\text{cr}}$ is [3, 3.3.11.]. The hope is that the composition $(h \circ - \circ \text{よ}) \circ q_!$ is easier to globalize than $\widetilde{\text{cr}}$ (otherwise the usage of the above 2-categorical tool is an unnecessary detour).
Indeed, both $(h \circ - \text{よ})$ and $q_!$ can be globalized into 2-functors, though for different reasons. For the first of the two, we can use the following result, referred to as "twisting by adjunction", which tells us how to globalizes (and generalize to any 2-category, not just that of categories) the statement that given an adjunction $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$, the functor $(L \circ - \circ R) : \text{Fun}(\mathcal{C}, \mathcal{C}) \to \text{Fun}(\mathcal{D}, \mathcal{D})$ is oplax monoidal.

Theorem 2.2.23. *([3, 2.1.16.]) Let* S *be a space,* $\boldsymbol{X}$ *be a 2-precategory and* $\phi : S \to \boldsymbol{X}_1^{\simeq}$ *a map of spaces. Denote the post compositions of* ϕ *by the source and target map respectively by* $\sigma, \tau : S \to \boldsymbol{X}_0$*. Suppose that for all* $a, b \in S$*, the pullback functor* $\phi(a)^* : \boldsymbol{X}(\tau(a), \tau(b)) \to \boldsymbol{X}(\sigma(a), \tau(b))$ *admits a left adjoint* $\phi(a)_!$*. Then there exists an oplax functor* $\phi^{tw} : \sigma^* \boldsymbol{X} \to \tau^* \boldsymbol{X}$ *such that for any* $a, b \in S$ *the functor* ϕ^{tw} *on mapping categories is the composite*

$$\boldsymbol{X}(\sigma(a), \sigma(b)) \xrightarrow{\phi(b)_*} \boldsymbol{X}(\sigma(a), \tau(b)) \xrightarrow{\phi(a)_!} \boldsymbol{X}(\tau(a), \tau(b)).$$

Remark 2.2.24. One particular case of the above result is when $\phi(a)$ is a left adjoint in the 2-precategory $\mathbf{X}$, with right adjoint $\phi(a)^R$. In this case, the various left adjoints we require to the pullbacks $\phi(a)^*$ automatically exist and are given by precomposition by $\phi(a)^R$,

The details of how to apply this result to the various Yoneda adjunctions $h \dashv \text{よ}$ are given by [3, 3.3.12.].

We now briefly discuss the globalization of $q_!$. For this we will use a different method for constructing 2-functors, arguably the most nave one. For a 2-precategory $\mathbf{X}$ and an object x of this category, there is a 2-functor represented by x, sending y to the category of maps $\mathbf{X}(x, y)$.

Remark 2.2.25. Notice that this is in fact not so simple as it may seem. Just like defining representable functors out of an $(\infty, 1)$-category, this requires an understanding of (in our case the $(\infty, 2)$-categorical version of) straightening/unstraightening. This is derived in [3, A.2.31.] from the work done in [28].

One can hope that since $q_!$ is quite a simple functor, it might be represented by quite a simple object. A very simple candidate is $*$ the category on 1 object and no non-trivial higher cells. So perhaps applying the nave idea of considering the functor corepresented by $*$ leads to a simple enough functor from which we could construct $q_!$. In a lucky twist[3], the functor thus constructed is exactly the desired globalization of $q_!$, as is studied in [3, 3.2.13.] and [3, 3.2.15.].

It turns out will only define formal multilinearization on the image of ι rather than on all formal symmetric sequences, which in order to post-compose the formal cross-effect is obviously sufficient. For convenience we record the image of ι in the following definition.

Definition 2.2.26. ([3, 3.3.15.]) Let $\mathcal{F}$ be the smallest subcategory of $\mathrm{Diff}_{\geq 1}^{\mathrm{F\,Sym}}$ for which each of the mapping categories $\mathcal{F}(\mathcal{C}, \mathcal{D})$ is the full subcategory of $\mathrm{SymFun}_{\geq 1}^{L}(\mathcal{P}^{\mathrm{fin}}(\mathcal{C}), \mathcal{P}^{\mathrm{fin}}(\mathcal{D}))$ on the image of ι.

Our summary of the process of defining a global formal multilinearization will be a fair bit shorter than that of defining the formal cross-effect, not because it is particularly simpler, but because it uses a single 2-category theory theorem. The hope stems from the fact that each $\widetilde{P_{\vec{1}}}$ is a composition of localizations, which are amongst some of the nicest functors. We can hope that if such a "nice" class of functors are suitably compatible, then they can be elegantly glued. These hopes are realized in the following.

Definition 2.2.27. ([3, A.3.22.] and [3, A.3.24.]) Let $\mathbf{X}$ be a 2-precategory and suppose we are given a full subcategory $\mathbf{X}' \subset \mathbf{X}$ such that for each pair of objects $x, y \in \mathbf{X}_0$, the inclusion $\mathbf{X}'(x, y) \to \mathbf{X}(x, y)$ admits a left adjoint $L_{x,y}$. We call the family of $\{L_{x,y}\}_{x,y \in \mathbf{X}_0}$ a *family of reflective localizations*.
We further call this family compatible if for any x, y and z in $\mathbf{X}_0$ and any $f \in \mathbf{X}(x, y)$ and $g \in \mathbf{X}(y, z)$ the maps

$$L_{x,z}(g \circ f) \to L_{x,z}(L_{y,z} g \circ f) \quad \text{and} \quad L_{x,z}(g \circ f) \to L_{x,z}(g \circ L_{x,y} f)$$

are equivalences.

Proposition 2.2.28. *([3, A.3.25.]) Let $\boldsymbol{X}$ be a 2-precategory and suppose that we are given a compatible family of reflective localizations $\{L_{x,y}\}_{x,y \in \boldsymbol{X}_0}$ as in the previous definition. Then there is a 2-precategory $L\boldsymbol{X}$ with the same objects as $\boldsymbol{X}$ and whose higher cells are the $L_{\bullet,\bullet}$-local higher cells of $\boldsymbol{X}$. Composition in $L\boldsymbol{X}$ is given by*

$$\circ^L : L\boldsymbol{X}(x, y) \times L\boldsymbol{X}(y, z) \to L\boldsymbol{X}(x, z), g \circ^L f = L(g \circ f).$$

where $\circ$ denotes composition in $\boldsymbol{X}$, and the units of $L\boldsymbol{X}$ are $L_{x,x}(\boldsymbol{1}_x)$. Furthermore, there is a lax functor $L\boldsymbol{X} \to \boldsymbol{X}$ and the family of reflective localizations $\{L_{x,y}\}_{x,y \in \boldsymbol{X}_0}$ assemble into a strong functor $L : \boldsymbol{X} \to L\boldsymbol{X}$ locally left adjoint to the lax functor $L\boldsymbol{X} \to \boldsymbol{X}$.

Remark 2.2.29. The result is in fact somewhat stronger than what we cited, as it also comes with a description of how to construct $L\mathbf{X}$ using straightening/unstraightening ideas, which we have omitted as we judged that it went too much into $(\infty, 2)$-categorical nitty gritty.

It therefore remains to verify that the various $L_{\mathcal{C},\mathcal{D}} := L \circ P_{\vec{1}} : \mathcal{F}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$ indeed assemble into a compatible family of reflective localization and carefully understand what applying the above theorem then yields to obtain the following desired result.

Proposition 2.2.30. *([3, 3.3.17.]) There exists a strong 2-functor of 2-precategories*

$$\widetilde{P_{\vec{1}}} : \mathcal{F} \to \mathrm{Diff}_{\geq 1}^{\mathrm{mlin}},$$

where the codomain is the 2-category of differentiable category and mapping categories given by

$$\mathrm{Diff}_{\geq 1}^{\mathrm{mlin}}(\mathcal{C}, \mathcal{D}) = \mathrm{SymFun}_{\omega, \geq 1}^{\mathrm{mlin}}(\mathcal{C}, \mathcal{D}).$$

The functor $\widetilde{P_{\vec{1}}}$ is the identity on objects and is given by mapping categories by $L_{\mathcal{C},\mathcal{D}} : \mathcal{F}(\mathcal{C}, \mathcal{D}) \to$ $\mathrm{SymFun}^L(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D}))$

[3]Perhaps, what makes $*$ non-trivial in our setting, and thus the connection with $q_!$ believable, is that we are considering $*$ as an object of Pr^{sym} and not of Pr^L. In the former category, $*$ is actually a free commutative and cofree cocommutative algebra, and so the presence of some non-trivial phenomena is perhaps less surprising than it was at first.

We will not provide the full details of the proof of the above theorem from the 2-categorical result on compatible localizations. Namely, we include the following result, showing that n-excisive approximations satisfy the condition which make them a compatible family of localization, but we do not discuss how the consideration of "formal multilinearization" enters into the picture.

Proposition 2.2.31. *([3, 3.1.33.]) Suppose $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors between differentiable categories and let $n \geq 1$. Then*

1. *if F is reduced the canonical map $P_n(GF) \to P_n((P_nG)F)$ is an equivalence;*
2. *if G is finitary, the canonical map $P_n(GF) \to P_n(G(P_n(F)))$ is an equivalence.*

In particular, if F is reduced and G is finitary, there is an equivalence

$$\partial_n(GF) \to \partial_n((P_nG)(P_nF))$$

Since we expect the "globalized" derivatives to be post composition by formal multilinearization of formal cross-effect, we expect the codomain of the former to be the codomain of the derivatives. In the way that we stated our theorems giving us the two functors in consideration, this is simply not the case. Luckily, the following result fixes this.

Proposition 2.2.32. *[3, 3.3.19.] There exists an equivalence of 2-categories*

$$Pr^{\mathrm{Sym}}_{St,\geq 1} \xrightarrow{\sim} \mathrm{Diff}^{\mathrm{mlin}}_{\geq 1}$$

which sends a stable presentable category to itself and which is the identity on mapping categories. In particular, composition of 1-cells agree.

Putting all of this together, up to the above equivalence, we can define ∂_* as $\widetilde{P_{\vec{1}}} \circ \widetilde{\mathrm{cr}}$ where $\widetilde{P_{\vec{1}}}$ and $\widetilde{\mathrm{cr}}$ are (respectively strong and lax) 2-functors. Thus defining the derivatives as a lax 2-functor, which is what we set out to do in theorem 2.2.18.

2.2.3 The stable chain rule

To move from the lax chain rule to a full chain rule, we need to show that the comparison map provided by the lax structure is an equivalence. This is only the case stably, in general one needs to replace the composition product of functor symmetric sequences with a somewhat more elaborate construction. As we only need the stable chain rule, we will not say a further word about the general case. So what we want to prove is the following theorem.

Theorem 2.2.33. *([3, 3.4.1.]) Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be a pair of composable reduced finitary functors between differentiable categories. If $\mathcal{D}$ is stable, the comparison map*

$$\mu : \partial_*(G) \circ \partial_*(F) \to \partial_*(GF)$$

provided by the lax structure on ∂_* is an equivalence.

The next theorem shows that up to the simple computation that, when $\mathcal{D}$ is stable, $\partial_*(\mathrm{Id}_{\mathcal{D}})$ is the identity symmetric sequence, the above local theorem is enough to promote the lax structure of the derivatives to a strong 2-functor structure when restricting to stable presentable differentiable categories.

Lemma 2.2.34. *([3, A.3.13.]) Let $F : \boldsymbol{X} \to \boldsymbol{Y}$ be a lax functor. Then F is a strong functor if and only if for any pair of composable 1-cells f, g and any object x the comparison maps $Fg \circ Ff \to F(gf)$ and $\boldsymbol{1}_{F(x)} \to F(\boldsymbol{1}_x)$ are equivalences.*

We will now work through a large amount of the proof of the above statement, in particular exploring the various reductions which take place, but saying nothing of technical details and the final computation, for which the author has nothing to add to the exposition of [3]. To start, we can work one arty at a time, i.e. we may study $\mu_n : (\partial_*(G) \circ \partial_*(F))_n \to \partial_n(GF)$. By proposition 2.2.31 the right vertical leg in the following square is an equivalence

$$\begin{tikzcd}
(\partial_* G \circ \partial_* F)_n \arrow[r] \arrow[d] & \partial_n(G \circ F) \arrow[d] \\
(\partial_* P_n G \circ \partial_* P_n F)_n \arrow[r] & \partial_n(P_n G \circ P_n F)
\end{tikzcd}$$

and the left vertical leg is an equivalence as the arity n-terms of a composition product of two sequences without anything in arity 0, depends only on the arity k part of both sequences, with $k \leq n$. In our case, for the arity k part is the same for the domain and the codomain as we have an equivalence between the kth derivative of F and of $P_n F$ since $k \leq n$. So it suffices in the above diagram to show that the bottom arrow is an equivalence in order to show that the top one is.

Next we show that we may assume F is k-homogeneous for some k. To this end, we show that $\partial_*(-) \circ \partial_* F$ and $\partial_*(- \circ F)$ preserve finite limits. Since the derivatives functor preserves finite limits, it will suffices to show that both composition of functors and functor symmetric sequences commute with finite limits in the left variable. For functor composition, this follows immediately from objectwise computation of limits in functor categories. For composition of functor symmetric sequences, since we are only considering reduced functors, the formula proven in proposition 2.2.15 can be used to understand $\partial_*(-) \circ \partial_* F$. Now because derivatives live in a stable category the coproduct appearing in the formula of proposition 2.2.15 is also a product, thus the result follows from objectwise computation of limits in functor categories and commutativity of limits amongst themselves.

Thanks to this, using naturality of μ we get the following comparison of fiber sequences (which are an example of a finite limit)

$$\begin{tikzcd}
\partial_*(D_k G) \circ \partial_* F \arrow[r] \arrow[d] & \partial_*(P_k G) \circ \partial_* F \arrow[r] \arrow[d] & \partial_*(P_{k-1} G) \circ \partial_* F \arrow[d] \\
\partial_*(D_k G \circ F) \arrow[r] & \partial_*(P_k G \circ F) \arrow[r] & \partial_*(P_{k-1} G \circ F).
\end{tikzcd}$$

The derivatives exist in a stable category, so that if the two vertical legs on the edges are equivalences, the same holds for the middle vertical leg. So by induction, we may assume G is k-homogeneous for some $k \leq n$. In this case, we know that G factors through the stabilization of $\mathcal{E}$ via $\Omega^\infty : \mathrm{Sp}(\mathcal{E}) \to \mathcal{E}$ by [23, 6.1.2.29.]. In summary. we have reduced to

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G'} \mathrm{Sp}(\mathcal{E}) \xrightarrow{\Omega^\infty} \mathcal{E}$$

with F n-excisive and G' k-homogeneous. The following lemma, by 2 out of 3, shows that in fact we may assume that $\mathcal{E}$ is stable.

Lemma 2.2.35. *([3, 3.4.6.]) Let $\mathcal{C}$ and $\mathcal{D}$ be differentiable categories and let $G : \mathcal{C} \to \mathrm{Sp}(\mathcal{D})$ be a reduced finitary functor. Then the following comparison map from the lax structure on the derivatives is an equivalence*

$$\partial_*(\Omega^\infty) \circ \partial_*(G) \to \partial_*(\Omega^\infty G).$$

The next insight is that both the domain and codomain of $\mu : \partial_*(G) \circ \partial_*(F) \to \partial_*(GF)$ commute with finite totalizations in the F variable. The derivatives commute with all finite limits, so for the domain, preservation of limits amounts to showing that the composition product preserves finite totalizations in the second variable. Similarly for the codomain, it suffices to show that G being n-excisive preserves finite totalizations, and hence since limits are computed objectwise, the expression GF commutes with finite totalizations in the F variable.
The relation of the composition product with totalizations is given by the following result.

Proposition 2.2.36. *([3, 3.2.12.]) Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be presentable categories. The composition product*

$$\circ : \mathrm{SymFun}^L_{\geq 1}(\mathcal{D}, \mathcal{E}) \times \mathrm{SymFun}^L_{\geq 1}(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^L_{\geq 1}(\mathcal{C}, \mathcal{E}),$$

preserves colimits in the first variable and sifted colimits in the second. If $\mathcal{D}$ and $\mathcal{E}$ are further stable, then it also preserves arity wise finite totalizations in both variables.

As for the preservation of finite totalizations by G, we have the following result.

Proposition 2.2.37. *([6, 3.37.]) Let $G : \mathcal{D} \to \mathcal{E}$ be a functor between cocomplete stable categories which is n-excisive and finitary. Then F preserves totalizations which are m-skeletal for some m and all geometric realizations.*

At this point we are able to assume that G is k-homogeneous and that F is n-excisive for some k and n and we know that if F is a totalizations of a nicer class of functors for which the statement we wish to show is true, we are done. Since F is n-excisive, we can write

$$F \simeq P_n F \to P_{n-1} F \to \cdots \to P_1 F.$$

In the stable setting, there is a strengthening of the classical Dold-Kan which establishes an equivalence between (co)filtered objects and (co)simplicial objects. The string of composable arrows above exhibits F as cofiltered, and thus encourages us to use (the opposite of) the following version of the Dold-Kan theorem.

Theorem 2.2.38. *([23, 1.2.4.1.]) Let $\mathcal{C}$ be a stable category, then there is a (canonical) equivalence*

$$\mathrm{Fun}(N(\mathbb{Z}_{\geq 0}), \mathcal{C}) \simeq \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathcal{C}).$$

Furthermore, this equivalence is compatible with colimits, i.e. the colimit of a filtered object is the geometric realization of the corresponding simplicial object.

In order to apply the above theorem in our case requires a description of the map going from the LHS to the RHS. This does not appear explicitly in [23], but can be understood using [36, 6.2.] where it is shown that the we can factor $\mathrm{Fun}(N(\mathbb{Z}_{\geq 0}), \mathcal{C}) \to \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathcal{C})$ as

$$\mathrm{Fun}(N(\mathbb{Z}_{\geq 0}), \mathcal{C}) \to \mathrm{Ch}_{\geq 0}(\mathcal{C}) \to \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathcal{C}),$$

where the second arrow is (an ∞-categorical lift) of the classical Dold-Kan correspondence and the first sends a filtered object $\cdots \to X_i \to X_{i+1} \to \cdots$ to a chain complex

$$\overline{X}_0 \leftarrow \cdots \leftarrow \overline{X}_i \leftarrow \overline{X}_{i+1} \leftarrow \cdots$$

where $\overline{X}_n \simeq \Omega^n \operatorname{cof}(X_{i-1} \to X_i)$. In particular, the n-simplicies of a simplicial object coming from a filtered object is a direct sum of shifts of cofibers of successive terms of the filtered object. In our case, we get that G can be seen as the totalizations of finite (because our cofiltered object has only finitely many non-zero graded pieces) coproducts of homogeneous functors.

And so, using that $\mu_n : (\partial_*(G) \circ \partial_*(F))_n \to \partial_n(GF)$ commutes with finite totalizations in the F-variable, we see that it suffices to prove that μ_n is an equivalence when G and F are both homogeneous functors. At this point no further simplification is possible, and the end of the proof relies on an explicit understanding of μ_n provided by [3, 3.3.14.] and [3, 3.3.23.]. As announced in the introduction of this subsection, we have nothing to add to [3] for this final step, and so we leave end our overview of the stable chain rule here.

2.3 Koszul Duality in Goodwillie calculus

Having now suitably discussed the stable chain rule, we include the last ingredient for the result we want, which we recall is the algebraic statement relating Koszul duality to Goodwillie calculus.

Proposition 2.3.1. *([3, 4.3.3.]) Let $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ be reduced finitary functors between differentiable categories. Then the map*

$$\partial_*(GF) \to \operatorname{Tot} \partial_*(G\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty F)$$

obtained by applying ∂_* to the Bousfield-Kan resolution is an equivalence.

Proof. It follows by the natural equivalence of proposition 2.2.31 that we can assume that G is n-excisive. Then using that $- \circ F$ and $\partial_*(-)$ preserve finite limits, we get a map of fiber sequences

$$\begin{tikzcd} \partial_*(D_kG \circ F) \arrow[r] \arrow[d] & \partial_*(P_kG \circ F) \arrow[r] \arrow[d] & \partial_*(P_{k-1}G \circ F) \arrow[d] \\ \operatorname{Tot}\partial_*(D_k(G)\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty F) \arrow[r] & \operatorname{Tot}\partial_*(P_k(G)\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty F) \arrow[r] & \operatorname{Tot}\partial_*(P_{k-1}(G)\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty F) \end{tikzcd}$$

which shows that, since we are in the stable case, the middle map is an equivalence if and only if the two outer ones are, and thus we can assume that G is k-homogeneous for some k. We can now use the classification of such functors to writhe G as $H\Sigma^\infty$.
In this case, the map we are interested in becomes

$$\partial_*(H\Sigma^\infty F) \to \operatorname{Tot}\partial_*(H\Sigma^\infty\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet\Sigma^\infty F).$$

In particular, the counit $\epsilon : \Sigma^\infty\Omega^\infty \to \mathrm{Id}_{\mathcal{D}}$ provides an extra codegenracy, and because the totalization of a coaugmented cosimplicial object $X_\star \to X_\bullet$ is just $X_\star$, we obtain the desired statement. ☐

Using the stable chain rule, the right hand side of the arrow in the above proposition can be rewritten as $\operatorname{coBar}(\partial_*(G\Omega^\infty), \partial_*(\Sigma^\infty\Omega^\infty), \partial_*(\Sigma^\infty F))$. Renaming G to K and considering the specific case that F is the identity, we obtain, using linearity of Σ^∞, the following result.

Corollary 2.3.2. *There is an equivalence*

$$\partial_*(K) \simeq \operatorname{coBar}(\partial_*(K\Omega^\infty), \partial_*(\Sigma^\infty\Omega^\infty), \boldsymbol{1}).$$

We now briefly recall some results which allow us to interpret the above statement as a Koszul duality statement.

In brief, Koszul duality, also called the bar/cobar adjunction, is the idea that under sufficient niceness assumptions, in the context of higher algebra, algebraic objects and coAlgebraic ones should be related. Striving to be as categorically minded as possible, we expect this relation to appear upon studying adjunctions which could relate Algebraic objects to coAlgebraic objects.
We might start by wondering whether we can recall an obvious adjunction between $\mathrm{Alg}(\mathcal{C})$ and $\mathrm{coAlg}(\mathcal{C})$ for some monoidal category $\mathcal{C}$. If it were so simple, we would be happy, but sadly this is not the case. So we might hope to pass by an intermediary category, the most obvious choice being $\mathcal{C}$ itself. Under some niceness assumptions on $\mathcal{C}$ and the tensor product, we have adjunctions $\mathrm{Free} : \mathcal{C} \leftrightarrows \mathrm{Alg}(\mathcal{C}) : U$ and $U : \mathrm{coAlg}(\mathcal{C}) \leftrightarrows \mathcal{C} : \mathrm{Cofree}$, and therefore composing them we might wish to study the pair of functors $\mathrm{Free} \circ U : \mathrm{CoAlg}(\mathcal{C}) \leftrightarrows \mathrm{Alg}(\mathcal{C}) : \mathrm{Cofree} \circ U$. These functors however are not related in any obvious way, and furthermore, it is unclear how generally we can expect to have both free algebras and cofree coAlgebras.
However, the idea of passing by an intermediary category is still an interesting one, and furthermore it is hard to imagine a nicer intermediary category than $\mathcal{C}$ itself. We focus on the algebraic half of the story. In order to explore other possible relations between $\mathcal{C}$ and some category of algebraic structures in $\mathcal{C}$, we consider as our Algebraic category $\mathrm{Lmod}_A(\mathcal{C})$ where A is some Algebra. Just as before, we might study a free/forgetful adjunction, but in case A is augmented, we have another option.

Definition 2.3.3. ([3, 4.1.10.]) Let $\mathcal{C}$ be a monoidal category and let $\epsilon : A \to \mathbf{1}$ be an augmented algebra. Then restriction of scalars induces a functor

$$\mathrm{triv}_A : \mathcal{C} \simeq \mathrm{Lmod}_{\mathbf{1}}(\mathcal{C}) \to \mathrm{Lmod}_A(\mathcal{C}).$$

In order to run through the idea outlined above to relate a category of algebraic structures to a category of coAlgebraic structures, we need to investigate whether the above functor admits an adjoint of some kind.

Proposition 2.3.4. *([3, 4.1.11.]) Let $\mathcal{C}$ be a monoidal category and let $\epsilon : A \to \mathbf{1}$ be an augmented algebra. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial diagrams. Then the functor* triv_A *admits a left adjoint, which we call* indec_A*.*

Proof. Let $\mathcal{E} \subset \mathrm{Lmod}_A(\mathcal{C})$ be the full subcategory on those objects M for which the functor

$$\mathrm{Map}_{\mathrm{Lmod}_A(\mathcal{C})}(M, \mathrm{triv}_A(-)) : \mathcal{C} \to \mathcal{S}$$

is corepresentable. Our goal can be rephrased as showing that $\mathcal{E}$ is the whole category of left A-modules. Indeed, we can consider the functor

$$\phi : \mathrm{Lmod}_A(\mathcal{C}) \to \mathcal{S}^{\mathcal{C}}$$

which sends M to the presheaf $\mathrm{Map}_{\mathrm{Lmod}_A(\mathcal{C})}(M, \mathrm{triv}_A(-))$. Our definition of $\mathcal{E}$ is saying that the restriction $\phi|_{\mathcal{E}}$ factors through the image of よ $: \mathcal{C} \to \mathcal{S}^{\mathcal{C}}$, which implies that in fact it factors through $\mathcal{C}$ since よ is a fully faithful embedding. So if $\mathcal{E} = onLmod_A(\mathcal{C})$, then ϕ gives a functor $\mathrm{Lmod}_A(\mathcal{C}) \to \mathcal{C}$, which upon post-composing with the Yoneda coembedding is naturally equivalent to $\mathrm{Map}_{\mathrm{Lmod}_A(\mathcal{C})}(M, \mathrm{triv}_A(-))$, which exactly says that ϕ is left adjoint to $\mathrm{triv}_A(-)$.

So our strategy consists in identifying some modules which are obviously in $\mathcal{E}$ and then using simplicial diagrams to notice that all modules must line in $\mathcal{E}$.
For modules of the form $A \otimes x$, for $x \in \mathcal{C}$, where the action comes from the multiplication of A (i.e. for free modules), we have the following

$$\mathrm{Map}_{\mathrm{Lmod}_A(\mathcal{C})}(A \otimes x, \mathrm{triv}_A(-)) \simeq \mathrm{Map}_{\mathcal{C}}(x, U \circ \mathrm{triv}_A(-)) \simeq \mathrm{Map}_{\mathcal{C}}(x, -),$$

which exactly shows that all free modules are in $\mathcal{E}$. This gives a partially defined left adjoint $F : \mathcal{E} \to \mathcal{C}$. Now, for an arbitrary left A-module M, consider the simplicial object $B_\bullet(A, A, M) = A^{\otimes(\bullet+1)} \otimes M$, whose geometric realization is M itself. Now, we can apply F degree wise to $B_\bullet(A, A, M)$, and then take geometric realization, which is a colimit, and so by universal property thereof, we see that $|F(B_\bullet(A, A, M))|$ is a suitable representative of $\mathrm{Map}_{\mathrm{Lmod}_A(\mathcal{C})}(M, \mathrm{triv}_A(-))$, thus showing that $\mathcal{E}$ is as big as we hoped and concluding the proof. □

Remark 2.3.5. The proof provides the explicit formula $\mathrm{indec}_A(M) = B(\mathbf{1}, A, M)$.

Now as we have an adjunction, we can hope for something coAlgebraic by considering the associated comonad $\mathrm{indec}_A \circ \mathrm{triv}_A$. Since we want this object to live in $\mathcal{C}$ rather than in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$, it will be interesting to evaluate this comonad at an object of $\mathcal{C}$ of particular interest. Arguably the simplest candidate is $\mathbf{1}$, which motivates the study of $B(\mathbf{1}, A, \mathbf{1})$ and the hope that this is coAlgebraic. For convenience, we will write this construction simply as $\mathrm{Bar}(A)$. The small bump prevents the heuristic we just described from being a proof is that $\mathrm{indec}_A \circ \mathrm{triv}_A$ is coAlgebraic with respect to composition of functors, but we want $\mathrm{Bar}(A)$ to be so with respect to the monoidal product in $\mathcal{C}$. Which is why we need the following result, which makes reference to the contents of appendix A.

Proposition 2.3.6. *([3, 4.1.12.] and [3, 4.1.13.]) Let $\mathcal{C}$ be a monoidal category and let $\epsilon : A \to \mathbf{1}$ be an augmented Algebra. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial diagrams. Then* $\mathrm{Bar}(A)$ *is the coendomorphism object of* $\mathrm{triv}_A(\mathbf{1})$*, where we use the standard right tensoring of* $\mathrm{Lmod}_A(\mathcal{C})$ *over* $\mathcal{C}$*.*
Furthermore, $\mathrm{Bar}(A)$ *is augmented.*

Proof. The proof uses that restriction of scalars is $\mathcal{C}$-linear, this can be understood as a case of commutativity of left and right actions, and is given precisely in [3, B.3.3.]. The first half of the result then follows directly from the adjunction, as we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(\mathrm{indec}_A(\mathrm{triv}_A(\mathbf{1})), x) &\simeq \mathrm{Map}_{\mathrm{LMod_A}}(\mathrm{triv}_A(\mathbf{1}), \mathrm{triv}_A(x)) \\ &\simeq \mathrm{Map}_{\mathrm{LMod_A}}(\mathrm{triv}_A(\mathbf{1}), \mathrm{triv}_A(1) \otimes \mathbf{x}).\end{aligned}$$

this is exactly definition A.1.1, thus verifying the desired claim.
For the second half of the statement, by [3, 4.1.9.] (a specialization of which we recalled as proposition A.1.7), it suffices to find a $\mathcal{C}$-linear functor $\mathrm{LMod}\, A(\mathcal{C}) \to \mathrm{LMod}_{\mathbf{1}(\mathcal{C})} \simeq \mathcal{C}$. The forgetful functor does the trick, thus concluding the proof. □

All of what we have said dualizes to coaugmented coAlgebras $\eta : \mathbf{1} \to Q$. We summarize this in the following omnibus theorem.

Theorem 2.3.7. *Suppose $\mathcal{C}$ is a monoidal category which admits totalizations of cosimplicial objects and let $\eta : \mathbf{1} \to Q$ be an augmented coAlgebra. Corestriction of scalars gives a functor $\mathrm{triv}_Q : \mathcal{C} \to \mathrm{RcoMod}_Q(\mathcal{C})$ which admits a right adjoint prim_Q.*
This right adjoint admits an explicit formula $\mathrm{prim}_Q(N) = C(N, Q, \mathbf{1})$. We write $\mathrm{coBar}(Q)$ for $\mathrm{prim}_Q(\mathbf{1})$.
The object $\mathrm{coBar}(Q)$ is the endomorphism object of $\mathrm{triv}_Q(\mathbf{1})$, and thus admits an Algebra structure. Furthermore, this Algebra can be augmented.

We call $\mathrm{Bar}(A)$ the Koszul dual of A and similarly call $\mathrm{coBar}(Q)$ the Koszul dual of Q. The nature of $\mathrm{Bar}(A)$ and $\mathrm{coBar}(Q)$ as (co)endomorphism objects makes map into/out of them correspond to certain actions, and one can hope by a commutativity of left and right actions that we can relate $\mathrm{Bar}(A)$ and $\mathrm{coBar}(Q)$ by saying that maps $A \to \mathrm{coBar}(Q)$ or $\mathrm{Bar}(A) \to Q$ classify a compatible pair of an action and a coaction.
This is made precise in the following result.

Proposition 2.3.8. *([3, 4.1.14.]) Let $\mathcal{C}$ be a monoidal category admitting realization of simplicial diagrams and totalizations of cosimplicial diagrams. Let $\epsilon : A \to \mathbf{1}$ be an augmented algebra and $\eta : \mathbf{1} \to Q$ be a coaugmented coAlgebra. Then the spaces $\mathrm{Map}_{\mathrm{coAlg}^{\mathrm{coAug}}(\mathcal{C})}(\mathrm{Bar}(A), Q)$ and $\mathrm{Map}_{\mathrm{Alg}^{\mathrm{Aug}}(\mathcal{C})}(A, \mathrm{coBar}(Q))$ are both naturally equivalent to*

$$\mathrm{triv}_A(\mathbf{1}) \times_{\mathrm{LMod}_A(\mathcal{C})} \mathrm{LMod}_A(\mathrm{RMod}_Q(\mathcal{C})) \times_{\mathrm{RMod}_Q(\mathcal{C})} \mathrm{triv}_Q(\mathbf{1})$$

Remark 2.3.9. At first glance there is an odd asymmetry in that we expect that if $\mathrm{Map}_{\mathrm{coAlg}^{\mathrm{coAug}}(\mathcal{C})}(\mathrm{Bar}(A), Q)$ is equivalent to $\mathrm{triv}_A(\mathbf{1}) \times_{\mathrm{LMod}_A(\mathcal{C})} \mathrm{LMod}_A(\mathrm{RMod}_Q(\mathcal{C})) \times_{\mathrm{RMod}_Q(\mathcal{C})} \mathrm{triv}_Q(\mathbf{1})$, then the mapping space $\mathrm{Map}_{\mathrm{Alg}^{\mathrm{Aug}}(\mathcal{C})}(A, \mathrm{coBar}(Q))$ should be equivalent to $\mathrm{triv}_A(\mathbf{1}) \times_{\mathrm{LMod}_A(\mathcal{C})} \mathrm{RMod}_Q(\mathrm{LMod}_A(\mathcal{C})) \times_{\mathrm{RMod}_Q(\mathcal{C})} \mathrm{triv}_Q(\mathbf{1})$. I.e. One expects that running identical proofs for the two mapping spaces of interest, should swap LMod_A and RMod_Q in the end result. Since the actions are on different sides, it is not surprising that this does not matter, but ∞-categorically, this does require some work to be done carefully, undertaken as [3, B.3.1.]

From the above result, we deduce the following, that although we investigated a relation between Algebra and coAlgebra through modules and comodules, an adjunction between Algebraic structures and coAlgebraic structures arises, as desired.

Corollary 2.3.10. *([3, 4.1.16.]) Let $\mathcal{C}$ be a monoidal category admitting realization of simplicial diagrams and totalizations of cosimplicial diagrams. Then we have an adjunction*

$$\mathrm{Bar} : \mathrm{Alg}^{\mathrm{Aug}}(\mathcal{C}) \leftrightarrows \mathrm{coAlg}^{\mathrm{coAug}}(\mathcal{C}) : \mathrm{coBar}\,.$$

The final piece of the puzzle to interpret corollary 2.3.2 as a Koszul duality statement, is whether modules over A and comodules over $\mathrm{Bar}(A)$ can be related, which might be expected given that the bar/cobar adjunction was built up from understanding (co)modules over (co)augmented (co)Algebras.

The way this is undertaken is to compare the bar/cobar adjunction we have been discussing thus far to the one developed in [23, 5.2.], so that results proved for the latter can be used without worry. Since this would lead us too far astray, we state the following result without further support.

Proposition 2.3.11. *([3, 4.1.22.] from [5, 3.26.]) Suppose that $\mathcal{C}$ is a monoidal category such that tensor products commute with geometric realizations. Then there is a diagram*

$$\begin{tikzcd} \mathrm{LMod}^{\mathrm{Aug}}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}"] \arrow[d, "U_1"'] & \mathrm{LcoMod}^{\mathrm{Aug}}(\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}"] \arrow[d, "U_2"] \\ \mathrm{Alg}^{\mathrm{Aug}}(\mathcal{C}) \arrow[r, shift left, "\mathrm{Bar}"] & \mathrm{coAlg}^{\mathrm{coAug}}(\mathcal{C}) \arrow[l, shift left, "\mathrm{coBar}"] \end{tikzcd}$$

where both rows are adjunctions with the left adjoint on top, such that considering only left or right adjoints,the diagram commutes. If A is an augmented algebra, and M a left A-module, then Bar *sends M to to the* $\mathrm{Bar}(A)$*-comodule* $\mathrm{indec}_A(M) = B(\boldsymbol{1}, A, M)$*. Similarly if Q is a coaugmented coAlgebra, and N a Q-comodule,* coBar *sends N to the* $\mathrm{coBar}(Q)$*-module* $\mathrm{prim}_Q(N) = C(\boldsymbol{1}, Q, N)$*.*
Furthermore, $\mathrm{Bar} : \mathrm{LMod}^{\mathrm{aug}}(\mathcal{C}) \to \mathrm{LcoMod}^{\mathrm{aug}}(\mathcal{C})$ *is right $\mathcal{C}$-linear and sends U_1-coCartesian edges to U_2-coCartesian edges.*

In the same way that we refer to $\mathrm{Bar}(A)$ and $\mathrm{coBar}(Q)$ as Koszul dual to A and Q respectively, we use this terminology also for $\mathrm{Bar}(M)$ and $\mathrm{Cobar}(N)$, calling these Koszul dual, potentially specifying "relative to A or Q", to M or N respectively. There is obviously a version of the above result for right modules instead of left modules, for which we use the same vocabulary.
And so with this in hand, we can rephrase corollary 2.3.2 as follows.

Corollary 2.3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be differentiable categories, with $\mathcal{D}$ stable and $K : \mathcal{C} \to \mathcal{D}$ be a reduced finitary functor. Then the derivatives of K are the Koszul dual of the derivatives of $K\Omega^\infty_{\mathcal{C}}$ relative to $\partial_*(\Sigma^\infty_{\mathcal{C}}\Omega^\infty_{\mathcal{C}})$.*

Koszul duality is at its most powerful when the bar/cobar adjunction is an equivalence, and in fact it is in this case that it really deserves to be called a duality rather than just an adjunction. There is one specific situation where this is the case, and in fact it is the case of interest of us, as the result following the next definition shows.

Definition 2.3.13. ([3, 4.2.1.]) A symmetric sequence $F : \mathrm{Sym}(\mathcal{C}) \to \mathcal{C}$ is strongly positive if $F_0 = 0$ and $F_1 = \mathrm{Id}_{\mathcal{C}}$. Let $\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C})$ denote the full subcategory of $\mathrm{SymFun}^L(\mathcal{C}, \mathcal{C})$ on the strongly positive symmetric sequences.

Remark 2.3.14. Notice that the monoidal structure on symmetric functor sequences restricts to one one on strongly positive functor sequences, as this subcategory is closed under the composition product and contains the unit symmetric sequence. Further notice that (co)Algebras are automatically (and uniquely) (co)augmented in this case, by inclusion of/projection onto the arity 1-component.

Theorem 2.3.15. *([3, 4.2.4.]) The bar/cobar adjunction gives an adjoint equivalence*

$$\mathrm{Bar} : \mathrm{Alg}(\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C})) \leftrightarrows \mathrm{coAlg}(\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C})) : \mathrm{coBar}\,.$$

We do not prove the above statement, but provide the proof of the following related statement, as the proof is very similar, but has not appeared in the literature. We thank Thomas Blom for pointing this out.

Proposition 2.3.16. *Given an algebra $A \in \mathrm{Alg}(\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C}))$, the bar/cobar adjunction gives an equivalence*

$$\mathrm{Bar} : \mathrm{RMod}_A(\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C})) \leftrightarrows \mathrm{RcoMod}_{\mathrm{Bar}(A)}(\mathrm{SymFun}^L_+(\mathcal{C}, \mathcal{C})) : \mathrm{coBar}\,.$$

Remark 2.3.17. A dual statement working with a coAlgebra $Q \in \mathrm{coAlg}(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ also holds.

To prove the above statement requires the two following lemmas.

Lemma 2.3.18. *([3, 4.2.5.]) Let $Q \in \mathrm{coAlg}(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ be a coAlgebra, let M be a right and N a left Q-comodule such that $M_0 = N_0 = 0$. Then the arity n component of the cosimplicial symmetric sequence $C^\bullet(M, Q, N)$ $(n-1)$-coskeletal. In particular, the cobar construction is arity-wise a finite totalization.*

Lemma 2.3.19. *([3, 4.2.6.]) Let $A \in \mathrm{Alg}(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ and let $M \in \mathrm{RMod}_A(\mathrm{SymFun}^L(\mathcal{C},\mathcal{C}))$ be a module concentrated in a single arity $n \geq 1$. Then M is a trivial module.*

Proof of proposition 2.3.16. The proposition we are trying to prove amounts to showing that the unit η and counit ϵ of the $\mathrm{indec}_A \vdash \mathrm{prim}_{BA}$ adjunction are equivalences for all M. We only treat the unit, as the counit is sufficiently similar. For the remainder of the proof we abbreviate $\mathrm{Bar}(A) \simeq \mathrm{Bar}(\mathbf{1}, A, \mathbf{1})$ to BA.
Observe that the inclusion of symmetric sequences M such that $M_k = 0$ for all $k \geq n$, called n-truncated, into the the ∞-category of symmetric sequences admits an obvious two sided adjoint $\tau_{\leq n}$. Clearly $\lim_n \tau_{\leq n} M \simeq M$.
Also observe that $\lim_n : \mathrm{Bar}(\tau_{\leq n} M, A, \mathbf{1}) \to \mathrm{Bar}(M, A, \mathbf{1})$ is an equivalence, as in every arity this limit is eventually constant. In particular, the map

$$\eta : M \to \mathrm{prim}_{BA}\, \mathrm{indec}_A\, M$$

is the limit of the corresponding maps for $\tau_{\leq n} M$. Therefore, it suffices to show that $\eta : \tau_{\leq n} M \to \mathrm{prim}_{BA}\, \mathrm{indec}_A\, \tau_{\leq n} M$ is an equivalence for each n. This opens the gateway to proof by induction, since the claim is obviously true for $n = 1$.
Before proving this, there is some preliminary structure of $\mathrm{RMod}_A(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ and $\mathrm{RcoMod}_{BA}(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ that we need to establish. First notice that by proposition 2.2.36 the composition product preserves all colimits in the left variable. Since we are working with positive symmetric sequences the formula from proposition 2.2.15 applies, and furthermore since $\mathcal{C}$ is stable, the finite coproduct appearing in said formula is also a product. It follows then by objectwise computation of limits of functors that the composition product of symmetric sequences preserves limits in the left variable. It then follows from [23, 4.2.3.3.] and [23, 4.2.3.5.] that $\mathrm{RMod}_A(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ and $\mathrm{RcoMod}_{BA}(\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C}))$ have all limits and colimits that $\mathrm{SymFun}^L_+(\mathcal{C},\mathcal{C})$ has, and that the forgetful functor both preserves reflects these. In particular, it immediately follows that these categories are stable, and that the functors indec_A and prim_{BA} being adjoints are exact.
We therefore consider the following ladder diagram whose vertical legs are components of the unit of $\mathrm{indec}_A \vdash \mathrm{prim}_{BA}$ and whose rows are the obvious fiber sequence

$$\begin{tikzcd}
M_n \arrow[r] \arrow[d] & \tau_{\leq n} M \arrow[r] \arrow[d] & \tau_{\leq n-1} M \arrow[d] \\
\mathrm{coBar}(\mathrm{Bar}(M_n, A, \mathbf{1}), BA, \mathbf{1}) \arrow[r] & \mathrm{coBar}(\mathrm{Bar}(\tau_{\leq n} M, A, \mathbf{1}), BA, \mathbf{1}) \arrow[r] & \mathrm{coBar}(\mathrm{Bar}(\tau_{\leq n-1} M, A, \mathbf{1}), BA, \mathbf{1}).
\end{tikzcd}$$

Since this is a map of fiber sequences in a stable category, by induction, it suffices to prove η is an equivalence for the A-module M_n concentrated in degree n. Notice that by lemma 2.3.19, we are in particular considering η for a trivial module.
Since the module structure on M_n is trivial, that is to say we have an equivalence of modules $M_n \simeq M_n \circ \mathbf{1}$, we have an equivalence

$$B_\bullet(M_n, A, \mathbf{1}) \simeq M_n \circ B_\bullet(\mathbf{1}, A, \mathbf{1})$$

of simplicial objects. Now by proposition 2.2.36, the composition product preserves geometric realization in the second variable, so the above equivalence implies $\mathrm{Bar}(M_n, A, \mathbf{1}) \simeq M_n \circ \mathrm{Bar}(\mathbf{1}, A, \mathbf{1})$ as right BA-modules. We are dealing with finite totalizations by lemma 2.3.18, so since the composition product commutes with finite totalizations in the second variable, we get

$$\mathrm{coBar}(M_n \circ \mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathbf{1}) \simeq M_n \circ \mathrm{coBar}(\mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathbf{1}).$$

The map $\eta : M_n \to M_n \circ \mathrm{coBar}(\mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathbf{1})$ comes from a coaugmentation of the cosimplicial object $M_n \circ \mathrm{coBar}(\mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathrm{Bar}(\mathbf{1}, A, \mathbf{1}), \mathbf{1})$, which implies it is an equivalence. This shows the desired claim. $\square$

3 K-theory

3.1 Additive invariants

Algebraic topology can be succinctly summarized as studying functors from categories of "shapes" to algebraic categories, which for the remainder of this heuristic we simply call invariants. Accepting the study of invariants as a worthwhile one, it is of paramount importance to find invariants which are in some sense simple or elementary, so that they are still computable, but still retain a sufficient amount of information, so that computing these is not pointless. One of the first such examples is the Euler characteristic. We include the following definition for the reader's convenience.

Definition 3.1.1. Let X be a a finite CW-complex, then we define

$$\chi(X) = \sum_{k \in \mathbb{N}} (-1)^k |k\text{-cells of } X|.$$

Remark 3.1.2. Our first goal in this section is a light hearted motivation of K-theory as an Euler characteristic of categories. We therefore omit many details which would lead us astray. The interested reader can begin further inquiry on the nlab's page of the Euler characteristic [26]

This is arguably a very simple example of an invariant, as sufficiently low dimensional version of it can be explained to 10 year olds. And already with this definition, we can witness the power of algebraic topology, as for example it can prove the following a priori non obvious result.

Proposition 3.1.3. *Suppose (on an infinite flat plane) there are three houses A, B and C, and opposite to them, there are three utilities a, b and c. Then it is impossible to draw a line from each house to each utility with without these lines intersecting.*

This is a common toy problem, which when phrased as a question can be oddly frustrating. But it is in fact a not too distant corollary of the following result, which can easily be proven by induction.

Proposition 3.1.4. *If Γ is a planar graph (whose cellular structure includes the faces delimited by the edges), i.e. a set of vertices and edges which can be drawn in $\mathbb{R}^2$ without any of the edges overlapping, then $\chi(\Gamma) = 1$.*

Inspired by this, we might hope to study something akin to an Euler-characteristic in situations where we stretch the definition of shape quite far. One way this is done is by the following definition, which allows us to extend what we consider as a shape to be a "dualizable object in a monoidal category".

Definition 3.1.5. Let $\mathcal{C}$ be a monoidal category and X a dualizable object in $\mathcal{C}$. Then the Euler characteristic of X is the trace of the identity of X.

A key property of many different Euler characteristics is an additivity of some kind, seen for instance in the following result

Theorem 3.1.6. *[29, 1.3] Let $\mathcal{C}$ be a stably symmetric monoidal category with unit $\mathbb{1}$ and let $X \to Y \to Z$ be a fiber/cofiber sequence of dualizable objects. Then in $\pi_0(\mathrm{End}(\mathbb{1}))$ we have $\chi(Y) = \chi(X) + \chi(Z)$.*

At this point, one can wonder whether we can obtain an invariant as powerful as the Euler characteristic for some category of categories. There are several different generalities at which this can be done, and different paths one can take. Inspired by the historical trajectory of K-theory, the definition of a "global Euler characteristic" in Steimle's paper [34, 0.1.] and limited by the author's understanding, we consider the above theorem to be the core property that one wants, if something ought to be called an Euler characteristic. Ideally, one would want to define an Euler characteristic of categories on a class of categories which are dualizable in some sense. This avenue is pursued by Effimov in [10], but is beyond the scope of this project, so we will instead limit ourselves to (suitably small) stable categories. For convenience, we recall here the definition of the category of categories we will be working with.

Definition 3.1.7. Let $\mathrm{Cat}^{\mathrm{ex}}$ be the subcategory of Cat of stable categories and exact functors between them, i.e. those which preserve finite limits and finite colimits.

Since the category of stable categories is not itself stable, in making the above theorem the defining property of Euler characteristics for stable categories, which we will hence call additive invariants, one needs to clear up what sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ are "sent to a sum".

Definition 3.1.8. ([7, A.2.1.]) A pair of composable morphisms $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ in $\mathrm{Cat}^{\mathrm{ex}}$ with vanishing composite is called a *Verdier sequence* if it is both a fiber and cofiber sequence in $\mathrm{Cat}^{\mathrm{ex}}$. In this case, we call f a *Verdier inclusion* and p a *Verdier projection*.

Verdier sequences will not be exactly those which we demand are "sent to a sum" by additive invariants, but before going on to define those sequences, we indulge in a very elementary study of Verdier sequences. To this end, we introduce Verdier quotients, which will make studying Verdier sequences, inclusions and projection more manageable.

Definition 3.1.9. ([7, A.1.4.]) Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor of stable categories, we say that a map in $\mathcal{D}$ is an equivalence modulo $\mathcal{C}$ if its fiber, or equivalently its cofiber, lies in the smallest stable category containing the image of $\mathcal{C}$. We write $\mathcal{D}/\mathcal{C}$ for the localization of $\mathcal{D}$ with respect to W the collection of equivalences modulo $\mathcal{C}$ and we call $\mathcal{D}/\mathcal{C}$ the *Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$.

To understand Verdier quotients, we have the following result.

Proposition 3.1.10. *([7, A.1.5.]) Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then $\mathcal{D}/\mathcal{C}$ is stable, the localization functor $\mathcal{D} \to \mathcal{C}$ is exact, and the sequence*

$$\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$$

is a cofiber sequence in $\mathrm{Cat}^{\mathrm{ex}}$*. Furthermore, for every $x, y \in \mathcal{D}$, the map*

$$\mathrm{colim}_{[\beta : z \to y] \in \mathcal{C}_{/y}} \mathrm{Map}_{\mathcal{D}}(x, \mathrm{cof}(\beta)) \to \mathrm{Map}_{\mathcal{D}/\mathcal{C}}([x], [y])$$

is an equivalence.

As the sequences we are interested in are also fiber sequences, we need to understand the kernel of the map $\mathcal{D} \to \mathcal{D}/\mathcal{C}$, so that we have a chance at understanding when the sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a Verdier sequence. To this end, we have the following result.

Proposition 3.1.11. *([7, A.1.8.]) Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor of stable categories. The kernel of $p : \mathcal{D} \to \mathcal{D}/\mathcal{C}$ consists of all objects which are retracts of objects in the essential image of $f : \mathcal{C} \to \mathcal{D}$.*

With the above results in hand, we may give equivalent characterizations of Verdier inclusions and projection which do not require fitting them in a Verdier sequences.

Proposition 3.1.12. *([7, A.1.7]) Let $p : \mathcal{D} \to \mathcal{E}$ be an exact functor of stable categories. Then the following are equivalent:*

(i) The functor p is a Verdier projection.

(ii) The induced map $\mathcal{D}/\ker(p) \to \mathcal{E}$ is an equivalence.

(iii) The functor p is a localization at the maps it sends to equivalences.

Proof. The statement that p is a Verdier projection is the same as requiring the sequence $\ker(p) \to \mathcal{D} \to \mathcal{E}$ to be a Verdier sequence. Since it is already a fiber sequence, it suffices to verify that it is a cofiber sequence. By proposition 3.1.10, this amounts to verifying that the map $\mathcal{D}/\ker(p) \to \mathcal{E}$ is an equivalence. This gives the equivalence between the first two points.
Now if (ii) holds, the fact that $\mathcal{D} \to \mathcal{D}/\ker(p)$ is a localization at the map it sends to equivalences, implies the same holds of the map $p : \mathcal{D} \to \mathcal{E}$. Finally, if p is a localization at the map it sends to equivalences, using p is exact so preserves fiber sequences and since in the stable setting a map is an equivalence if and only if its kernel is 0, it follows that an arrow is sent to an equivalence by p if and only if the fiber of the arrow is in the kernel of p. Therefore, p is exactly a localization at the equivalences modulo p, which readily shows (ii). $\square$

Proposition 3.1.13. *([7, A.1.9.]) Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then the following are equivalent:*

(i) The functor f is a Verdier inclusion.

(ii) The induced map $\mathcal{C} \to \ker(\mathcal{D} \to \mathcal{D}/\mathcal{C})$ is an equivalence.

(iii) The functor f is fully faithful and its essential image is closed under retracts.

Proof. The functor f is a Verdier inclusion precisely if the sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a Verdier sequence, which is the case exactly if $f : \mathcal{C} \to \mathcal{D}$ is a kernel. The equivalence between (ii) and (iii) follows directly from proposition 3.1.11 and the fact that the kernel of a map p is the inclusion of the full subcategory of $\mathrm{dom}(p)$ on those objects which p sends to 0. $\square$

We can put the above two results together to get

Corollary 3.1.14. *([7, A.1.10.]) For a sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ with vanishing composite, the following are equivalent:*

(i) The sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ is a Verdier sequence.

(ii) The functor f is fully faithful with essential image closed under retracts in $\mathcal{D}$ and p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.

(iii) The functor p is a localization at the arrows it sends to equivalences, and f exhibits $\mathcal{C}$ as the kernel of p.

With all of this preliminary work in hand, we are finally ready to define the sequences which we will require additive invariants to turn into sums.

Definition 3.1.15. ([7, A.2.1.]) We call a Verdier sequence $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ *left (resp. right) split* if p admits a left (resp. right) adjoint. We call a Verdier sequence split if p admits both adjoints.

It is perhaps somewhat frustrating that the notion of (left/right) split is defined solely in terms of p. This can be remedied by the following result.

Proposition 3.1.16. *([7, A.2.8.]) Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{ex}}$ with vanishing composite. Then the following are equivalent:*

(i) The sequences is a left (resp. right) split Verdier sequence.

(ii) The sequence is a fiber sequence, and p admits a fully faithful left (resp. right) adjoint q.

(iii) The sequence is a cofiber sequence, f is fully faithful and admits a left (resp. right) adjoint g.

Furthermore, when these equivalent conditions are met, the sequence $\mathcal{E} \xrightarrow{q} \mathcal{D} \xrightarrow{g} \mathcal{C}$ obtained by passing to left (resp. right) adjoints is a right (resp. left) split Verdier sequence.

Before defining additive invariants precisely there is one more question which needs solving. For a symmetric monoidal category, we know the codomain of the Euler characteristic to be $\mathrm{End}(\mathbf{1})$, but since we are defining our "Euler characteristic of stable categories" only by analogy, it is perhaps not completely obvious what we want the codomain of additive invariants to be. Continuing to pursue the analogy, we might consider what is the closest to a monoidal unit in the category of stable categories. Motivated by the tensor product of presentable categories as considered in [23, 4.8.1.], in particular in light of [23, 4.8.1.23.], which says that Sp is the monoidal unit for the tensor product of stable presentable categories, we might venture replacing $\mathbf{1}$ by the category Sp of spectra for the codomain of additive invariants.
That is we want to somehow define additive invariants to be certain functors $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{End}(\mathrm{Sp})$. In order for the tensor product of presentable categories to be a worthy motivation for our choice of

codomain, we would ideally like to be working with presentable categories. Starting with $\mathrm{Cat}^{\mathrm{ex}}$, the most natural way to access presentable categories is to Ind-complete. This lands us in Pr^L_{st}, the category of presentable stable categories and cocontinuous functors between them. Therefore, the appropriate codomain for additive invariants should be $\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$.
Hopefully, with this final piece of motivation in hand, the reader is convinced of the value of the following definition.

Definition 3.1.17. An *additive invariant* is a functor $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ which sends split Verdier sequences to fiber/cofiber sequences. The collection of these assemble into a category $\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^{\mathrm{ex}}, \mathrm{Sp})$.

Remark 3.1.18. The attentive reader will remark that a weakness in our analogy is that we have had to replace the "sends certain sequences to sums" by "sends certain sequences to fiber/cofiber sequences". Something of this sort was to be expected in light of wanting to understand more than π_0 of additive invariants.

For convenience, we also record here the definition of localizing invariants.

Definition 3.1.19. A *localizing invariant* is a functor $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ which sends Verdier sequences to fiber/cofiber sequences. The collection of these assemble into a category $\mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{ex}}, \mathrm{Sp})$.

3.2 The universal example

As it currently stands, it could be that there are no non-trivial additive invariants. In this section, we focus on arguably the most famous additive invariant, which is universal "under the groupoid core", called algebraic K-theory. We recall to the reader the definition of the groupoid core functor.

Definition 3.2.1. The groupoid core functor $(-)^{\simeq} : \mathrm{Cat} \to \mathcal{S}$, also written as $\iota : \mathrm{Cat} \to \mathcal{S}$ is the functor which sends an ∞-category to its maximal subgroupoid.

It would take far too much work to construct K-theory ex-nihilo and to provide its universal property. We decide to accept what is done in [4], in particular the two results which we will recall below, at which point we derive what is usually meant by the universal property of K-theory as is done in Saunier's thesis [33].

The first result from [4] gives us a perspective on additive invariants where they are simply cocontinuous functors out of a special "category of motives", which is meant to replace $\mathrm{Cat}^{\mathrm{ex}}$ by the version of this category which additive invariants know about.

Theorem 3.2.2. *([4, 6.6],[4, 6.7] and [4, 6.10.]) Denote by U_{add} the following composite*

$$\mathrm{Cat}^{\mathrm{ex}} \xrightarrow{\mathrm{Idem}(-)} \mathrm{Cat}^{\mathrm{perf}} \xrightarrow{よ} \mathcal{P}((\mathrm{Cat}^{\mathrm{perf}})^{\omega}) \xrightarrow{\gamma} \mathcal{M}^{\mathrm{un}}_{\mathrm{add}} \xrightarrow{\Sigma^\infty} \mathcal{M}_{\mathrm{add}}.$$

This composite is an additive invariant, which preserves filtered colimits, and is universal in the sense that, for any stable $\mathcal{D}$, it induces an equivalence

$$U^*_{\mathrm{add}} : \mathrm{Fun}^L(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{add}}_{\omega}(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{D}).$$

We denote the inverse of this equivalence by $\widetilde{(-)}$.

Remark 3.2.3. Since we will not need the details of the construction of U_{add}, we tolerate not going in depth in the various notation appearing in the above result, but we do want to say just enough for the interested reader to not start reading [4] at a complete loss.

- A category is called idempotent complete if all idempotents are split. The interested reader can start reading further at [22, 4.4.5.].
- The category $\mathrm{Cat}^{\mathrm{perf}}$ is the category of idempotent complete stable categories and exact functors between them.

- The functor $\mathrm{Idem}(-)$ is idempotent completion.
- The functor γ is a localization with a respect to the set of maps $よ(\mathcal{D})/よ(\mathcal{C}) \to よ(\mathcal{E})$ whenever $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a split Verdier sequence.
- We technically only defined additive functors when $\mathcal{D} = \mathrm{Sp}$ as for 99% of this project the extra generality is not helpful. In fact the only place we will use this generality in an important way is in this section. So we say it here in passing, for stable $\mathcal{D}$, a functor $\mathrm{Cat}^{\mathrm{ex}} \to \mathcal{D}$ is an additive invariant when it sends split Verdier sequences to (co)fiber sequences.
- The category $\mathcal{M}_{\mathrm{add}}$ is the stabilization of the category of $\mathcal{M}^{\mathrm{un}}_{\mathrm{add}}$.

One can think of U_{add} as the functor algebraic K-theorist would study if we did not have a bias towards studying spectra. But since we do, we would like to get a functor to Sp built from U_{add}, which using that $\mathcal{M}_{\mathrm{add}}$ is stable can be done by considering mapping spectra.

Theorem 3.2.4. *([4, 7.13.]) There is a natural equivalence of spectra*

$$\hom_{\mathcal{M}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^\omega), U_{\mathrm{add}}(-)) \simeq K(-).$$

In particular, the cocontinuous functor $\widetilde{K} : \mathcal{M}_{\mathrm{add}} \to \mathrm{Sp}$ *is corepresentable by* $U_{\mathrm{add}}(\mathrm{Sp}^\omega)$*, which is therefore a compact object of* $\mathcal{M}_{\mathrm{add}}$*.*

Proof. The first part is the content of [4, 7.13.], whereas the second part follows from the previous theorem, since the isomorphism $\hom_{\mathcal{M}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^\omega), U_{\mathrm{add}}(-)) \simeq K(-)$ implies that $\hom_{\mathcal{M}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^\omega), U_{\mathrm{add}}(-))$ is an additive invariant, and so corresponds to a cocontinuous functor $\mathcal{M}_{\mathrm{add}} \to \mathrm{Sp}$. This functor can immediately be seen to be the one corepresented by $U_{\mathrm{add}}(\mathrm{Sp}^\omega)$. And so the equivalence given by [4, 7.13.] corresponds under the equivalence of the previous theorem to the desired corepresentabillity statement. $\square$

Remark 3.2.5. The way we introduced things, we are almost taking the above result as a definition of K-theory, and so the reader who is interested in this document as isolated from the rest of the literature ought to ignore the presence of a proof of the above statement.

Proposition 3.2.6. *([33, 4.9.]) The natural transformation* $\Sigma^\infty_+ \iota \to K$ *of functors* $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ *exhibits algebraic K-theory as the initial finitary additive invariant under* $\Sigma^\infty_+\iota$*, the post-composition by* $\Sigma^\infty : \mathcal{S} \to \mathrm{Sp}$ *of the groupoid core.*

For the proof we will need the following three lemmas.

Lemma 3.2.7. *The groupoid core functor* $\iota : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{S}$ *is representable by* $\mathrm{Sp}^{\mathrm{fin}}$*.*

Proof. This follows from the fact that the mapping space $\mathrm{Hom}_{\mathrm{Cat}^{\mathrm{ex}}}(\mathrm{Sp}^{\mathrm{fin}}, \mathcal{C})$ is by definition the groupoid core of the ∞-category $\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Sp}^{\mathrm{fin}}, \mathcal{C})$. But this category of exact functors is equivalent to $\mathcal{C}$, with the equivalence given by evaluation at S^0. $\square$

Lemma 3.2.8. *The left Kan extension of* $\mathrm{Map}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathcal{S}$ *along* $f : \mathcal{C} \to \mathcal{D}$ *is* $\mathrm{Map}_{\mathcal{D}}(f(X), -)$*.*

Proof. The functor $f_!(\mathrm{Hom}_{\mathcal{C}}(X, -))$ is the initial functor $F : \mathcal{D} \to \mathcal{S}$ with a map $\mathrm{Hom}_{\mathcal{C}}(X, -) \Rightarrow F \circ f$. An obvious candidate is $\mathrm{Hom}_{\mathcal{D}}(f(X), -)$ with the map being the obvious one

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(f(X), f(Y)).$$

To see this is initial, use the ∞-category version of the Yoneda lemma[4] to see that $\mathrm{Nat}_{\mathcal{C}\to\mathcal{S}}(\mathrm{Hom}_{\mathcal{C}}(X, -), F \circ f) \simeq F(f(X)) \simeq \mathrm{Nat}_{\mathcal{D}\to\mathcal{S}}(\mathrm{Hom}_{\mathcal{D}}(f(X), -), F)$. $\square$

Lemma 3.2.9. *([33, 5.23.]) Let* $\mathcal{C}$ *be a stable category and* $F : \mathcal{C} \to \mathrm{Sp}$ *an exact functor. The natural transformation* $\eta : \Sigma^\infty_+\Omega^\infty F \to F$ *exhibits its target as the initial exact functor under its source.*

[4] By which we mean [22, 5.1.5.2.]

Proof. Because we have an adjunction $\Sigma^\infty_+ \vdash \Omega^\infty$, for any exact functor $G : \mathcal{C} \to \mathrm{Sp}$ we have a commutative diagram

$$\begin{tikzcd} \mathrm{Nat}(F,G) \arrow[r, "\Omega^\infty"] \arrow[d, "\eta^*"'] & \mathrm{Nat}(\Omega^\infty F, \Omega^\infty G) \arrow[d, "\mathrm{Id}"] \\ \mathrm{Nat}(\Sigma^\infty_+\Omega^\infty F, G) \arrow[r, "\simeq"] & \mathrm{Nat}(\Omega^\infty F, \Omega^\infty G). \end{tikzcd}$$

What we wish to show is that the vertical map is an equivalence, for that it will suffice to show that the top arrow is so. This follows because $\mathcal{C}$ is stable and therefore we have an equivalence

$$(\Omega^\infty)_* : \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathrm{Sp}) \to \mathrm{Fun}^{\mathrm{rex}}(\mathcal{C}, \mathcal{S}).$$

In particular, this functor is fully faithful, giving the desired result. □

Proof of proposition 3.2.6. By theorem 3.2.2, we get the following commutative (because the top arrow is a restriction/corestriction of the bottom one) diagram, where the bottom arrow is an equivalence

$$\begin{tikzcd} \mathrm{Fun}_\omega(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \arrow[r, "U^*_{\mathrm{add}}"] & \mathrm{Fun}_\omega(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{D}) \\ \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \arrow[r, "\simeq"] \arrow[u, hook] & \mathrm{Fun}^{\mathrm{add}}_\omega(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{D}) \arrow[u, hook] \end{tikzcd}.$$

All of these functors have left adjoints, resulting in the diagram

$$\begin{tikzcd} \mathrm{Fun}_\omega(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) \arrow[d, "P_1"'] & \mathrm{Fun}_\omega(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{D}) \arrow[l, "U_{\mathrm{add},!}(-)"'] \arrow[d, "(-)_{\mathrm{add}}"] \\ \mathrm{Fun}^{\mathrm{L}}(\mathcal{M}_{\mathrm{add}}, \mathcal{D}) & \mathrm{Fun}^{\mathrm{add}}_\omega(\mathrm{Cat}^{\mathrm{ex}}, \mathcal{D}) \arrow[l] \end{tikzcd}.$$

where P_1 is the 1-excisive approximation from Goodwillie calculus, $(-)_{\mathrm{add}}$ is additivization which exists as it can be seen as a left localization at the universal K-equivalences[5] and $U_{\mathrm{add},!}(-)$ is Left Kan extension. Because the bottom arrow is an equivalence and by uniqueness up to natural equivalence of adjoints, we see that $(-)_{\mathrm{add}} \simeq P_1(U_{\mathrm{add},!}(-)) \circ U_{\mathrm{add}}$. Now it suffices to show that plugging in $\Sigma^\infty \iota$ the infinite suspension of the groupoid core, in the left hand side of this equivalence, yields K-theory. For this we use that the groupoid core ι is representable by $\mathrm{Sp}^{\mathrm{fin}}$ and that left Kan extension along $f : \mathcal{C} \to \mathcal{D}$ of $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is $\mathrm{Hom}_{\mathcal{D}}(f(X), -)$. Because Σ^∞_+ is a left adjoint, by commutativity of left adjoints and left Kan extension we get that the additivization of $\Sigma^\infty_+ \iota$ is $P_1(\Sigma^\infty_+ \mathrm{Hom}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), -)) \circ U_{\mathrm{add}}$. Now we use lemma 3.2.9 to see that $\hom(X, -)$ is the initial exact functor under $\Sigma^\infty_+\Omega^\infty \hom_{\mathcal{C}}(X, -)$, which can be rewritten as $\Sigma^\infty_+ \mathrm{Hom}_{\mathcal{C}}(X, -)$. Therefore our expression for the additivization of $\Sigma^\infty_+ \iota$ simplifies to

$$\hom_{\mathcal{M}_{\mathrm{add}}}(U_{\mathrm{add}}(\mathrm{Sp}^{\mathrm{fin}}), U_{\mathrm{add}}(-)).$$

Up to idempotent completion, this is K-theory by theorem 3.2.4. □

In the coming sections, we will need some further properties of K-theory. In particular, we need to know that the tools of Goodwillie calculus recalled in the previous section apply and our method to apprehend the derivatives of K-theory will use that $K : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ is in fact a localizing invariant, although it is only universal as an additive invariant. The only not yet established point of order, to in time investigate the Taylor tower of K-theory, is that $\mathrm{Cat}^{\mathrm{ex}}$ is a differentiable category. This is a consequence of [23, 6.1.1.9.], which explains how compactly generated implies differentiable, and the following result.

Proposition 3.2.10. *([4, 4.25.]) The category* $\mathrm{Cat}^{\mathrm{ex}}$ *is compactly generated.*

[5] See [1, 4.7.]

This is achieved by realizing this category as an accessible localization of an ∞-category constructed out of the nerve of a model category of categories enriched in Sp, a result which is motivated by the fact that the $\mathrm{Hom}(-,-)$ spaces of a stable category can be enhanced to $\mathrm{hom}(-,-)$-spectra. A handle on the model category of categories enriched in Sp is achieved in [4, 2.4.].

The fact that K-theory is a localizing invariant is captured by the following result.

Theorem 3.2.11. *([18, 6.1.]) The algebraic K-theory functor $K : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ is Verdier localizing.*

The proof uses an alternative, arguably more explicit, model for algebraic K-theory, which we know to be the same as the object we are studying as they share the same universal property [18, 5.1.]. From there, the author of [18] are able to bootstrap the additivity of K-theory to the stronger property of being localizing via [18, 2.9.], which gives conditions for an additive functor to be localizing.

3.3 Stabilizing the domain of K-theory

Considering K-theory as an object of study of Goodwillie calculus means understanding the calculus of functors $\mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$. However, this turns out to be uninteresting essentially because of the following result.

Proposition 3.3.1. *The suspension functor $\Sigma : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Cat}^{\mathrm{ex}}$ is the 0-functor.*

Proof. The suspension of $\mathcal{C}$ a stable category is the cofiber of the unique map $\mathcal{C} \to \mathbf{0}$, and therefore can be studied by proposition 3.1.10. The result then follows since $\mathbf{0}/\mathcal{C}$ is a localization of $\mathbf{0}$, which is a groupoid, and thus $\mathbf{0}/\mathcal{C} \simeq 0$. $\square$

This is quite disappointing, but there is a fix. Indeed, this result can be interpreted as saying that $\mathrm{Cat}^{\mathrm{ex}}$ is too complex for any global approximation to be fruitful. However, we can work locally around some category $\mathcal{C}$ and consider K-theory instead as a functor $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{Sp}$, for which we will, somewhat abusively, use the same notation as when viewing K-theory as a functor out of $\mathrm{Cat}^{\mathrm{ex}}$. In this case suspension is no longer trivial, as the 0-object is $\mathcal{C}$ and so for example $\Sigma_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}}(\mathbf{0} \to \mathcal{C})$ is $\mathcal{C} \oplus \mathcal{C}$. However, since $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ is not stable, before directly understanding the derivatives of K-theory, we will want to understand the stabilization of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$, then the derivatives of $K \circ \Omega^\infty$ and then we use corollary 2.3.12 to understand the derivatives of K-theory. Henceforth in this section, $\mathcal{C}$ denotes some object in $\mathrm{Cat}^{\mathrm{ex}}$.

To understand the stabilization of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ in full, we encourage the reader to consult [16], which is the main source for this section. We will content ourselves by briefly summarizing the result of interest to us in that paper. We first define what will eventually be the stabilization of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$.

Definition 3.3.2. The category $\mathrm{BiMod}(\mathcal{C})$ can be described in one of the following equivalent ways

(i) Biexact functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$.

(ii) Exact functors $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathrm{Sp}$.

(iii) Exact functors $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

(iv) Colimit preserving functors $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$.

The association $\mathcal{C} \mapsto \mathrm{BiMod}(\mathcal{C})$ is contravariantly functorial in $\mathcal{C}$ by restriction, so we can therefore unstraighten this construction yielding

Definition 3.3.3. ([16, 2.2]) The fibration $\pi : \mathrm{Cat}^{\mathrm{Lace}} \to \mathrm{Cat}^{\mathrm{ex}}$ is the unstraightening of the contravariant functor sending $\mathcal{C}$ to the category $\mathrm{BiMod}(\mathcal{C})$.

The above fibration can be thought of as a "global" version of stabilization, the precise term for which is stable envelope[6], as opposed to the "local" stabilization $\mathrm{Sp}(\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}})$, which we are trying to comprehend. In order to extract the desired local stabilizations, we wish to construct an adjunction over $\mathrm{Cat}^{\mathrm{ex}}$, whose fibers are the "$\Sigma^\infty \dashv \Omega^\infty$"-adjunctions of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$. This is achieved by the following result, whose components we spend the rest of the section detailing.

Theorem 3.3.4. *([16, 2.8] and [16, 2.9]) There is an adjunction $\widetilde{L} \dashv \widetilde{\mathrm{Lace}}$ which fit into a commutative diagram*

$$\begin{tikzcd}
(\mathrm{Cat}^{\mathrm{ex}})^{\Delta^1} \arrow[rr, bend left, "\widetilde{L}"] \arrow[rr, phantom, "\perp"] \arrow[dr, bend right, "\mathrm{cod}"'] & & \mathrm{Cat}^{\mathrm{Lace}} . \arrow[ll, bend left, "\widetilde{\mathrm{Lace}}"] \arrow[dl, bend left, "\pi"] \\
& \mathrm{Cat}^{\mathrm{ex}} &
\end{tikzcd}$$

This diagram exhibits the adjunction as relative to $\mathrm{Cat}^{\mathrm{ex}}$ *in the sense of [23, 7.3.2.2.], in particular* cod *and* π *are categorical fibrations.*
Furthermore, the diagram exhibits $\mathrm{Cat}^{\mathrm{Lace}}$ *as the stable envelope of* $(\mathrm{Cat}^{\mathrm{ex}})^{\Delta^1}$ *over* $\mathrm{Cat}^{\mathrm{ex}}$ *in the sense of [23, 7.3.1.1].*

There are many ingredients of theorem 3.3.4 which require further comment in order for the above result to specialize to the result of interest, the first of which is the following result and proceeding remark, clarifying to us what it means to have a "relative" adjunction.

Proposition 3.3.5. *([23, 7.3.2.5.]) Suppose we are given a commutative diagram of categories*

$$\begin{tikzcd}
\mathcal{C} \arrow[dr, "q"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "p"] \\
& \mathcal{E} &
\end{tikzcd}$$

such that q and p are categorical fibrations. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with $pF = q$ and $u : \mathrm{Id}_{\mathcal{C}} \to G \circ F$ a natural transformation exhibiting F as a left adjoint to G relative to $\mathcal{E}$. Then for every functor $\mathcal{E}' \to \mathcal{E}$, denoting by $F' : \mathcal{C} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{D} \times_{\mathcal{E}} \mathcal{E}'$ and $G' : \mathcal{D} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{C} \times_{\mathcal{E}} \mathcal{E}'$ the functors induced by F and G respectively, the induced natural transformation $u' : \mathrm{Id}_{\mathcal{C}'} \to G' \circ F'$ exhibits F' as a left adjoint to G' relative to $\mathcal{E}'$.

The next is that stable envelopes are stable under pullback and that the stable envelope of $\mathcal{C} \to \{*\}$ is $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$. This is contained in the following two results.

Proposition 3.3.6. *([23, 7.3.1.3]) Suppose we have a pullback of simplicial sets, with vertical legs presentable fibrations*

$$\begin{tikzcd}
\mathcal{C}_0 \arrow[r, "\alpha"] \arrow[d, "p_0"'] & \mathcal{C} \arrow[d, "p"] \\
\mathcal{D}_0 \arrow[r] & \mathcal{D}.
\end{tikzcd}$$

If $u : \mathcal{C}' \to \mathcal{C}$ is a stable envelope of p, then the pullback of u along α is a stable envelope of p_0.

Remark 3.3.7. When $\mathcal{E}$ is the trivial one-object category, a relative adjunction over $\mathcal{E}$ is just an ordinary adjunction. And so the following result can be interpreted as saying that a relative adjunction is a fiberwise adjunction. This can be seen by direct application of the first characterization of relative adjunction in [23, 7.3.2.2.].

We combine the above with the following key example of a stable envelope.

[6]Although we will use this term, we will only need basic properties of this notion, and not the full definition, thus we refrain from repeating the latter, but the interest reader can find it as [23, 7.3.1.1.]

Example 3.3.8. ([23, 7.3.1.14.]) Let $\mathcal{C}$ be a presentable category. Then the map $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is a stable envelope of $\mathcal{C} \to \{*\}$.

All of the above directly imply the following result, which is the one of interest to us.

Proposition 3.3.9. *The stabilization of* $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ *is the category* $\mathrm{BiMod}(\mathcal{C})$*. Under this identification* Σ^∞ *corresponds to* $L_\mathcal{C} : \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{BiMod}(\mathcal{C})$ *and* Ω^∞ *to* $\mathrm{Lace}(\mathcal{C}, -) : \mathrm{BiMod}(\mathcal{C}) \to \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$*, where both of these functors are the ones induced by* $\widetilde{L}$ *and* $\widetilde{\mathrm{Lace}}$ *upon pulling back along* $\{\mathcal{C}\} \to \mathrm{Cat}^{\mathrm{ex}}$.

As the functors $L_\mathcal{C}$ and $\mathrm{Lace}(\mathcal{C}, -)$ are directly induced by $\widetilde{L}$ and $\widetilde{\mathrm{Lace}}$, one might assume it worthwhile to define the latter directly. However, these definitions are quite involved and not of direct relevance to us, so we give directly the definitions of the functors which appear in the above proposition.

Definition 3.3.10. The functor $\mathrm{Lace}(\mathcal{C}, -) : \mathrm{BiMod}(\mathcal{C}) \to \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ sends a bimodule $M : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ to the category over $\mathcal{C}$ given by the following pullback

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d, "\gamma_M"'] & \mathrm{Ind}(\mathcal{C})^{\Delta^1} \arrow[d, "(\mathrm{dom},\mathrm{cod})"] \\
\mathcal{C} \arrow[r, "(よ, M)"] & \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C}).
\end{tikzcd}$$

Remark 3.3.11. This definition could have been more succinctly phrased as saying that $\mathrm{Lace}(\mathcal{C}, M)$ is the lax-equalizer of よ and M in the sense of [25, II.1.4.]. The point of this perspective is twofold. First of all, it gives us access to [25, II.1.5.], which gives a hold of mapping spaces in $\mathrm{Lace}(\mathcal{C}, M)$, helps us understand (co)limits in this category and makes it clear that this category is stable. Second, in light of the heuristic similarity between the Yoneda embedding and the identity functor, section II.5 of [25] encourages us to think of $\mathrm{Lace}(\mathcal{C}, M)$ as a category of coAlgebras for the functor M. We make this perspective precise[7] in section 4.3.

Remark 3.3.12. There is a small subtlety in why the construction $M \mapsto \mathrm{Lace}(\mathcal{C}, M)$ is functorial in M, as a priori a map $\alpha : M \to N$ yields a non-invertible 2-cell in $\mathrm{Fun}(\bullet \to \bullet \leftarrow \bullet, \mathrm{Cat}^{\mathrm{ex}})$, but $\mathrm{Lace}(\mathcal{C}, M)$ is a 1-categorical pullback. Therefore, one might fear that α could possibly not induce a map $\mathrm{Lace}(\mathcal{C}, M) \to \mathrm{Lace}(\mathcal{C}, N)$.
This problem is solved by [3, A.2.28], which says that colimits indexed by 1-categories in a 2-category $\mathcal{X}$ which admits cotensors by [1] can be computed in the underlying 1-category $\mathcal{X}^{\leq 1}$. So that functoriality of $\mathrm{Lace}(\mathcal{C}, -)$ follows from the fact that it is in fact a 2-categorical pullback.

Remark 3.3.13. Since $\mathcal{C}$ is stable it is in particular pointed, so that the map $\mathrm{Ind}(\mathcal{C})^{\Delta^1} \to \mathrm{Ind}(\mathcal{C}) \times \mathrm{Ind}(\mathcal{C})$ admits a section given by sending a pair of objects to the 0-arrow between them. By universal property of pullbacks this gives a natural map $\mathcal{C} \to \mathrm{Lace}(\mathcal{C}, M)$.

Definition 3.3.14. The functor $L_\mathcal{C} : \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{BiMod}(\mathcal{C})$ is given by sending a morphism $f : \mathcal{D} \to \mathcal{C}$ to the bimodule (in the perspective where these are biexact functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Sp}$) $(f^{\mathrm{op}} \times f)_! \operatorname{Hom}_\mathcal{D}(-, -)$.

Both for $L_\mathcal{C}$ and $\mathrm{Lace}(\mathcal{C}, -)$, we will at times indulge in suppressing $\mathcal{C}$ from the notation, hoping that it is clear from context.

3.4 The derivatives of $K \circ \Omega^\infty$

Although not yet present in full detail in the literature, there is enough for a succinct description of the derivatives of

$$K \circ \Omega^\infty_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}} : \mathrm{Bimod}(\mathcal{C}) \to \mathrm{Sp}$$

to make its way into this project. For convenience, we will simply write this as $K(\mathcal{C}, -)$. This functor is referred to as *laced K-theory*, though we will sometimes simply call it K-theory. All of our Goodwillie

[7] Barring some caveats which we make clear in the same section.

calculus in this section is strictly in the bimodule variable.

For the details of what we present in this section, we encourage the reader to consult [16], the second part of [30] and [33]. We will simply cite the results without having the opportunity to say much more about them, as this would go beyond the scope of this project.

The understanding of the derivatives of laced K-theory involves two pieces, the understanding of the first derivative, and the following result, whose notation we expand upon in the proceeding remark, which says that this is enough.

Theorem 3.4.1. *([33, 6.18.]) Let* $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ *be a finitary Verdier-localizing invariant. Then there is a Cartesian square*

$$\begin{array}{ccc} P_n E^{\mathrm{cyc}}(\mathcal{C}, M) & \longrightarrow & P_1 E^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{h}\,C_n} \\ \downarrow & & \downarrow \\ P_{n-1} E^{\mathrm{cyc}}(\mathcal{C}, M) & \longrightarrow & P_1 E^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})^{\mathrm{t}\,C_n}. \end{array}$$

In particular, the nth homogeneous part of E *is given by*

$$P_1 E^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})_{\mathrm{h}\,C_n}.$$

Remark 3.4.2.

- The tensor power which appears in the above result is in reference to the monoidal structure on $\mathrm{BiMod}(\mathcal{C})$ which comes from recalling that this category can be seen as a category of endofunctors, which comes equipped with a composition monoidal structure.
- We are using the same convention for $E \circ \Omega^{\infty}_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}}$ as for $K \circ \Omega^{\infty}_{\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}}$ which we explained in the beginning of this section, that is to say the variable $\mathcal{C}$ is fixed and Goodwillie calculus is undertaken in the second variable.
- The way in which the functor $E^{\mathrm{cyc}}(\mathcal{C}, -)$ relates to $E(\mathcal{C}, -)$ is the content of the beginning of subsection 3.4.2. The only necessary property of $E^{\mathrm{cyc}}(\mathcal{C}, -)$ needed to decipher the above statement is that $E^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})$ comes equipped with a C_n-action.
- The notations $(-)^{\mathrm{h}\,C_n}, (-)^{\mathrm{t}\,C_n}$ and $(-)_{\mathrm{h}\,C_n}$ are in reference to the homotopy fixed points, the homotopy Tate-construction and the homotopy orbits respectively. These are related by the definition of homotopy Tate construction as the cofiber of the natural map $(-)_{\mathrm{h}\,C_n} \to (-)^{\mathrm{h}\,C_n}$.

Remark 3.4.3. The detailed proof of the above statement has, at the time of writing, not yet appeared in the literature, but an informal explanation is present in Saunier's thesis. We refrain from repeating this explanation here as it was beyond the scope of this project to both understand and somewhat rephrase what appears in Saunier's thesis.

To obtain the second part of the result from the first, one must simply recall that as we are in the stable setting, the fiber of parallel maps in a pullback square are equivalent, and that cofiber sequence are automatically fiber sequences.

3.4.1 Trace theories

Our next goal is to give the context to understand the cyc superscript, and its point in understanding the first derivative of laced K-theory. We will only provide a very general overview introducing core ideas. So we need to work introducing the notion of a trace theory. The basic idea of a trace theory $T(A_0; \vec{C})$ is that it is a functor, which has two asymmetric inputs, the first input can be thought of as the algebra over which we are working, and the second is a finite list of (A_i, A_{i+1})-bimodules, with $A_{n+1} = A_0$, and we require T to have some cyclic invariance, i.e. we have equivalences $T(A; C_1, ..., C_n) \simeq T(A; C_2, ..., C_n, C_1)$. In particular there is a C_n action on $T(A; C, ..., C)$.

Remark 3.4.4. Trace theories get their name as they emulate the core property of the trace of matrices of cyclic invariance. In order to resemble the above more closely we can view the trace as a map $\Lambda_{\mathrm{Vec}_k} \to k$, where the domain is a class consists of a choice of vector spaces $V_0, V_1, \cdots, V_n$ and a finite "circle" of composable arrows $\alpha_1 : V_0 \to V_1, \alpha_2, V_1 \to V_2$ and $\alpha_0 : V_n \to V_0$. Then the cyclic invariance of the trace can be written as $tr(V_0; \alpha_0 \circ ... \circ \alpha_1) = tr(V_i; \alpha_{i+1} \circ ... \circ \alpha_i)$.

The rub is that ∞-categorically, the number of coherence issues which need to be dealt with to make this precise is quite high. For this we will need to carefully design a suitable domain of a trace theory.

Definition 3.4.5. ([30, 4.1.6.]) Let $\mathcal{C}$ be an ∞-category and $\mathbf{B}$ and $(\infty, 2)$-category. Then there is an ∞-category $\mathrm{Fun}_{\mathrm{colax}}(\mathcal{C}, \mathbf{B})$ whose objects are functors from $\mathcal{C}$ to $\iota_1\mathbf{B}$, the 1-category obtained from $\mathbf{B}$ by forgetting non-invertible 2-cells, and whose morphisms are colax natural transformations.

Remark 3.4.6. Heuristically, a colax natural transformation of two functors f, g is the data of a 1-cell $f(c) \to g(c)$ in $\mathbf{B}$ for each $c \in \mathcal{C}$ and a 2-cell

$$\begin{tikzcd} f(c_0) \arrow[r] \arrow[d] & g(c_0) \arrow[d] \\ f(c_1) \arrow[r] \arrow[ur, Rightarrow] & g(c_1) \end{tikzcd}$$

in $\mathbf{B}$ for each arrow $c_0 \to c_1$ in $\mathcal{C}$.

For technical reasons, we will want to restrict only to colax natural transformations for which the 1-cells are left adjoints, we write this subcategory as $\mathrm{Fun}^{ladj}_{\mathrm{colax}}(\mathcal{C}, \mathbf{B})$. The next ingredient we need towards a formal definition of the domain of a trace theory, is a way to consider a "finite circle of composable arrows". This will be achieved by unstraightening functors out of the following combinatorial category.

Definition 3.4.7. ([30, A.0.19.]) We define Λ to be the (non-full) subcategory of Cat_∞ spanned by the 1-categories free on finite cyclic directed graphs, and functors between them that induce degree 1 maps on geometric realizations.
Every object of Λ is isomorphic to the free category on the oriented graph $0 \to 1 \to \cdots \to n \to 0$ for some non-negative integer n, which we denote by $[n]_\Lambda$.

This allows us to define the eventual domain of trace theories as follows.

Definition 3.4.8. ([30, 4.1.9.]) Let $\mathbf{B}$ be an $(\infty, 2)$-category, then define $\Lambda_{\mathbf{B}} \to \Lambda^{\mathrm{op}}$ to be the coCartesian unstraightening of the functor $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(-, \mathbf{B}) : \Lambda^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

This grants a suitable 2-cell functoriality, despite the fact that $\Lambda_{\mathbf{B}}$ is an $(\infty, 1)$-category. Indeed, the objects of $\Lambda_{\mathbf{B}}$ lying over $[n]_\Lambda$ already involve the data of the 1-cells of $\mathbf{B}$, as they are composable circles of $(n+1)$-morphisms (with a specified order). We fix a notation for these.

Notation 3.4.9. Given a composable circle of maps in $\mathbf{B}$ $T_1 : \mathcal{C}_0 \to \mathcal{C}_1, T_2 : \mathcal{C}_1 \to \mathcal{C}_2, \cdots, T_0 : \mathcal{C}_n \to \mathcal{C}_0$, we write the corresponding object of $\Lambda_{\mathbf{B}}$ by $(\mathcal{C}_0; \vec{T})$ with $\vec{T} = (T_1, ..., T_0)$.

And so the 1-cells of $\Lambda_{\mathbf{B}}$ consist of 2-cells from $\mathbf{B}$. With this we can define trace theories.

Definition 3.4.10. ([30, 4.1.13.]) A trace theory on $\mathbf{B}$ with values in an ∞-category $\mathcal{E}$ is a functor $\Lambda_{\mathbf{B}} \to \mathcal{E}$ which inverts coCartesian edges with respect to the fibration $\Lambda_{\mathbf{B}} \to \Lambda$. These assemble into a category $\mathrm{TrThy}(\mathbf{B}, \mathcal{E})$.

Remark 3.4.11. To understand the previous definition, it is clearly useful to understand what the coCartesian edges over $C \to D$ are. These are given by the evident precomposition functors

$$\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(D, \mathbf{B}) \to \mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(C, \mathbf{B}).$$

Remark 3.4.12. Whenever we "forget" to specify what $\mathbf{B}$ we are working on, whether it is by suppressing it from the notation or omitting it from the conversation we are implicitly talking about the case $\mathbf{B} \simeq \mathrm{Pr}^L_{st,(\omega)}$ trace theories, where this is the category of compactly generated presentably stable categories, and cocontinuous functors between them.
In this case, we might denote $\Lambda_{\mathbf{B}}$ (and later $\Delta_{\mathbf{B}}$) simply by Λ_{st} (and later Δ_{st}).

Notice that if we forget the arrow from n to 0 of $[n]_\Lambda$, we recover $[n]$ the object of the simplex category Δ viewed as a subcategory of Cat, and furthermore this combinatorial observation in fact underlies a functor $\Delta \to \Lambda$. This suggest that pulling back the fibration $\Lambda_{\mathbf{B}} \to \Lambda$, yields a map $\Delta_{\mathbf{B}} \to \Lambda_{\mathbf{B}}$, and upon restriction might lead to an interesting notion of Δ-trace theories. Given the average reader's expected increased familiarity with Δ over Λ, we might want to investigate this further, hoping for good fortune.
It turns out that we can in fact motivate Δ-trace theories more convincingly than by appealing to familiarity, by the difference between $[0]$ and $[0]_\Lambda$. The difference being that the former is terminal, whereas the latter is not, worse still Λ has neither initial nor terminal object. This small categorical advantage that Δ has over Λ ends up compounding so that many results involving existence of (co)limits[8] end up working only for Δ-trace theories. With this motivation out of the way, we can properly define $\Delta_{\mathbf{B}}$ and Δ-trace theories.

Definition 3.4.13. ([30, 4.1.9.]) There is a functor $\bullet_\Lambda : \Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ along which we can restrict $\mathrm{Fun}^{\mathrm{ladj}}_{\mathrm{colax}}(-, \mathbf{B})$. Unstraightening this functor gives a coCartesian fibration $\Delta_{\mathbf{B}} \to \Delta^{\mathrm{op}}$.

Definition 3.4.14. ([30, 4.1.13.]) A Δ-trace theory on $\mathbf{B}$ with values in $\mathcal{E}$ is a functor $\Delta_{\mathbf{B}} \to \mathcal{E}$ which inverts coCartesian edges. These assemble into a category $\mathrm{TrThy}_\Delta(\mathbf{B}, \mathcal{E})$.

The two notions of trace theories are related by the following result.

Lemma 3.4.15. *([30, 4.1.23.]) Let $\boldsymbol{B}$ be an $(\infty, 2)$-category and $\mathcal{E}$ an ∞-category. There is an S^1-action on $\mathrm{TrThy}_\Delta(\boldsymbol{B}, \mathcal{E})$ and an equivalence*

$$\mathrm{TrThy}_\Delta(\boldsymbol{B}, \mathcal{E})^{hS^1} \simeq \mathrm{TrThy}(\boldsymbol{B}, \mathcal{E}).$$

Remark 3.4.16. One of the core theoretical features of trace theories is that $T(b, \mathrm{Id})$ comes equipped with an S^1-action (see [30, 4.1.21.] for a heuristic and [30, 4.2.1.] for a detailed treatment), which is an extension of the C_n action on $T(A; C, \cdots, C)$ which we alluded to right before remark 3.4.4.
This S^1-action, which we will not have the opportunity to discuss further, has nothing to do with the one appearing in the above result.

We are almost ready to introduce the superscript cyc. However, before going into this, there is one final technical subtlety we need to discuss. Our main object of interest is the functor $K : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ seen as a *Verdier*-localizing invariant, whereas in [30], the author is instead interested in non-connective K-theory, which is functor out of $\mathrm{Cat}^{\mathrm{perf}}$, the category of idempotent complete stable categories and exact functors between these. In this case, he writes K^{cn}, for what we just call K, and what he writes as K is then the universal *Karoubi*-localizing invariant. As the notation and terminology suggests, these are intimately related, as K^{cn} is just the connective part of K.
Nonetheless, we will continue to mean Verdier localizing when we just say localizing, and we will continue to mean connective K-theory even when we write K without any superscript[9].

[8] And since (co)limits, Kan extensions and adjoints all subsume each other, this observation extends to any result involving any of these as well.

[9] It is therefore imaginable that some of our results are technically false, and really do require working with Karoubi localizing invariants. To the best of the author's knowledge this does not happen, and in any case does not matter given the level of detail we are working with. It furthermore is an acceptable concession, as [30, 4.1.42.] confirms that at least as far theorem 3.4.44 is concerned, we can in fact indulge in this. And theorem 3.4.44 is the only result we actually need.

3.4.2 The $(-)^{\mathrm{cyc}}$-construction and its purpose

The definition of the functor $E \mapsto E^{\mathrm{cyc}}$ will require some auxiliary pieces, whose existence is obtained by explicit construction, but for increased digestibility given the intent of survey rather than detailed exposition, we will "define" these pieces as adjoints, via the following two results.

Proposition 3.4.17. *([30, 4.1.33.]) There is a functor* $\mathrm{End}_\omega : \Lambda_{st} \to \mathrm{Cat}^{\mathrm{ex}}$ *which when restricted along* $\Delta_{st} \to \Lambda_{st}$ *is right adjoint to the functor sending* $\mathcal{C}$ *to* $(\mathrm{Ind}(\mathcal{C}), \mathrm{Id})$ *lying over* $[0]$.

We can think of the functor End_ω as a generalization of $\mathrm{Lace}(\mathcal{C}, M)$ which takes as input a circle of composable arrows potentially on more than one object.

Proposition 3.4.18. *([30, 4.1.38.] and [30, 4.1.37.]) There is a functor* $\prod : \Lambda_{st} \to \mathrm{Cat}^{\mathrm{ex}}$ *which when restricted along* $\Delta_{st} \to \Lambda_{st}$ *is right adjoint to the functor sending* $\mathcal{C}$ *to* $(\mathrm{Ind}(\mathcal{C}), 0)$ *lying over* $[0]$. *Furthermore, there is a natural transformation* $\tau : \mathrm{End}_\omega \to \prod$ *which admits a natural section* σ.

The functor $\prod$ can be thought of as sending a circle of composable arrows to the product of $\mathcal{C}_i^\omega$ with the $\mathcal{C}_i$ running over the vertices objects appearing in the circle of composable maps.

With this in hand we can finally define the $(-)^{\mathrm{cyc}}$ construction. A major part of the remainder of this section will be spent on understanding how this construction relates localizing invariants and trace theories.

Definition 3.4.19. ([30, 4.1.39.]) Let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathcal{E}$ be a functor with $\mathcal{E}$ stable. We denote by $E^{\mathrm{cyc}} : \Lambda_{st} \to \mathcal{E}$ the functor sending a composable circle $(\mathcal{C}; \vec{T})$ to $\mathrm{fib}(E(\mathrm{End}_\omega(\mathcal{C}; \vec{T})) \to E(\prod(\mathcal{C}; \vec{T})))$.

Remark 3.4.20. It is a natural frustration given our choice of presentation that bound to this document there is no way to know explicitly what E^{cyc} is. As a consolation prize, we offer a heuristic shared to the author by Maxime Ramzi. When $\vec{T} = (T)$, i.e. we are considering a circle of composable arrows with only 1 arrow, one ought to think of $E^{cyc}(\mathcal{C}, T)$ as the part $E(\mathrm{Lace}(\mathcal{C}, T))$, which in turn ought to be thought of as a T-infinitesimal thickening of $\mathcal{C}$, which does not come from $E(\mathcal{C})$.
In the case of K-theory, this heuristic manifests as the equivalence $K^{\mathrm{cyc}}(\mathcal{C}, M) \simeq K(\mathrm{Lace}(\mathcal{C}, M))/K(\mathcal{C})$, where the map $K(\mathcal{C}) \to K(\mathrm{Lace}(\mathcal{C}, M))$ is K-theory of the map described in remark 3.3.13.

Remark 3.4.21. One might wonder why we are bothering with the somewhat involved construction of E^{cyc}, especially since in [16, 4.42.] the authors are able to identify the derivative in the M-variable of $K(\mathcal{C}; M)$ directly. The meta-reason for this is simply that we are following [30] for our computation of the first derivative of K-theory, and the path taken there involves K^{cyc}. The more mathematical reason is that we want to consider reduced functors, so that we are in the setting where the chain rule applies, and thus we have access to corollary 2.3.12.

The first result which actually justifies all the energy we have put into defining trace theories and E^{cyc} is the following.

Proposition 3.4.22. *([30, 4.1.40.]) Let* $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ *be a localizing invariant, then* $E^{\mathrm{cyc}} : \Lambda_{st} \to \mathrm{Sp}$ *is a trace theory.*

In fact, trace theories obtained this way are of a particular kind, the so called reduced trace theories, which we define next.

Definition 3.4.23. ([30, 4.1.44.]) A (Δ-)trace theory F is reduced if for any $(\mathcal{C}; \vec{T})$ for which there is a T_i which is the 0 morphism, then $E(\mathcal{C}; \vec{T}) \simeq 0$.

The point of noticing this extra structure is that it gives E^{cyc} a universal property, which will in due time be instrumental in understanding $P_1 K^{\mathrm{cyc}}(\mathcal{C}, -)$.

Corollary 3.4.24. *([30, 4.1.47.]) The map* $s : E \circ \mathrm{End}_\omega \to E^{\mathrm{cyc}}$ *is the initial map out of* $E \circ \mathrm{End}_\omega$ *to a reduced* Δ*-trace theory, where* s *is a section to the natural map* $E^{\mathrm{cyc}} \simeq \mathrm{fib}(E \circ \mathrm{End}_\omega \to E \circ \prod) \to E \circ \mathrm{End}_\omega$.

Remark 3.4.25. The map s exists because of general abstract nonsense that for a fiber sequence $F \to E \to B$, the map $F \to E$ admits a section if the map $E \to B$ admits a retract. In our case the section of $E \to B$ exists by the content of proposition 3.3.13.

Having said a couple of words on how to get trace theories from localizing invariants, it will be useful to know how to go the other way. For this we need the following definition, which will enable us to speak of localizing trace theories.

Definition 3.4.26. ([30, 4.1.49.]) Let $(\mathcal{C}, T) \xrightarrow{i} (\mathcal{D}, S) \xrightarrow{p} (\mathcal{E}, Q)$ be a sequence of morphisms in Λ_{st} (resp. Δ_{st}) lying over $[0]_\Lambda$ (resp. $[0]$). We call this a *laced Verdier sequence* if the underlying sequence of categories $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is a laced category and the induced sequence $iTi^R \to S \to p^R Q p$ is a (co)fiber sequence in $\mathrm{Fun}^L(\mathcal{D}, \mathcal{D})$.

Remark 3.4.27. A very basic example of such sequence are those of the form $(\mathcal{C}, \mathrm{Id}) \to (\mathcal{D}, \mathrm{Id}) \to (\mathcal{E}, \mathrm{Id})$, where $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ is an ordinary Verdier sequence. This is the content of [7, A.2.9.]

Definition 3.4.28. ([30, 4.1.51.]) Let $T : \Lambda_{st} \to \mathcal{E}$ (resp. $T : \Delta_{st} \to \mathcal{E}$) be a trace theory (resp. Δ-trace theory) with values in a stable category. We call such T localizing if it sends laced Verdier sequences to (co)fiber sequences in $\mathcal{E}$.
When we wish to restrict to the category of localizing (Δ-)trace theories, we indicate this with the superscript $^{\mathrm{loc}}$.

Applying the preceding remark to the above definition yields the following result.

Proposition 3.4.29. *([30, 4.1.53.]) Let $T : \Delta_{st} \to \mathrm{Sp}$ be a localizing Δ-trace theory. Then the functor sending $\mathcal{C}$ to $T(\mathrm{Ind}(\mathcal{C}), \mathrm{Id})$ is a localizing invariant $\mathrm{Cat}^{\mathrm{ex}}$.*
Furthermore, the association sending T to $T_{\mathrm{Id}} : \mathcal{C} \mapsto T(\mathcal{C}, \mathrm{Id})$ can be made into a functor.

One might hope that we are done, but disappointingly, for a localizing invariant E, the trace theory E^{cyc} is not localizing. There is nonetheless a relation between $E \mapsto E^{\mathrm{cyc}}$ and $T \mapsto T_{\mathrm{Id}}$ captured in the following result.

Corollary 3.4.30. *([30, 4.1.58.]) Let $E : \mathrm{Cat}^{\mathrm{ex}} \to \mathrm{Sp}$ be a localizing invariant and T a reduced trace theory. We then have an equivalence*

$$\mathrm{Map}_{\mathrm{TrThy}_\Delta(\mathrm{Sp})}(E^{\mathrm{cyc}}, T) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{Cat}^{\mathrm{ex}}, \mathrm{Sp})}(E, T_{\mathrm{Id}}).$$

Proof. This follows directly from the universal property of E^{cyc} and the adjunction of proposition 3.4.17. □

The attentive reader will wonder why the word "adjunction" is missing from the above result. This is because with our assumptions on T, T_{Id} need not be a localizing invariant. We might then try to restrict to localizing trace theories, but we still will not get an adjunction as E^{cyc} need not be a localizing trace theory. We can fix this with the following stronger notion.

Definition 3.4.31. ([30, 4.1.55.]) A reduced trace theory T is called exact if for each $\mathcal{C} \in \mathrm{Cat}^{\mathrm{ex}}$ the functor $T(\mathcal{C}, -) : \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \to \mathcal{E}$ is exact. When we wish to specify we are working in the category of exact trace theories, we do this with the superscript $^{\mathrm{ex}}$.

We capture the fact that this is a stronger notion in the following result, which we leave unproven.

Proposition 3.4.32. *([30, 4.1.54.]) An exact trace theory is localizing.*

Now if we wish for exact trace theories to relate to localizing invariants by an adjunction, like we almost had with localizing trace theories, we need a canonical way to make a trace theory exact. The fact that we have such a tool is the key technical advantage of exact trace theories over localizing trace theories. The reason trace theories admit exact approximations is by an application of Goodwillie calculus.

Proposition 3.4.33. *([30, 4.1.60.] and [33, 5.26.]) For $\mathcal{E}$ stable and differentiable, the inclusion* $\mathrm{TrThy}^{\mathrm{ex}}_{\Delta}(\mathcal{E}) \to \mathrm{TrThy}^{\mathrm{ex}}(\mathcal{E})$ *admits a left adjoint, which we denote by P_1. Over* $[0]$, *$P_1 T(\mathcal{C}, M)$ is given by $P_1 T(\mathcal{C}, -)(M)$, i.e. we are considering Goodwillie calculus in the second variable.*

One might hope that the above result is trivial once we have the exact approximation of $T(\mathcal{C}, -) : \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \to \mathcal{E}$ one $\mathcal{C}$ at a time. But in fact, bundling all the $P_1 T(\mathcal{C}, -)$ together does take some work, which would lead us to far astray to properly delve into. With this in hand, we have the following result, which suitably relates exact trace theories and localizing invariants, by simply combining the universal properties of all the constituent pieces.

Corollary 3.4.34. *([30, 4.1.61.]) Let $\mathcal{E}$ be a stable differentiable category. We have an adjunction*

$$P_1(-^{\mathrm{cyc}}) : \mathrm{Fun}^{\mathrm{loc}}(\mathrm{Cat}^{\mathrm{ex}}, \mathrm{Sp}) \leftrightarrows \mathrm{TrThy}^{\mathrm{ex}}_{\Delta}(\mathrm{Sp}) : \bullet_{\mathrm{Id}}.$$

With this result in hand we are able to give a powerful universal characterization of the exact trace theory $P_1(K^{\mathrm{cyc}})$.

Proposition 3.4.35. *([30, 4.1.62.]) As a functor of exact trace theories, evaluation at* $(\mathrm{Sp}, \mathrm{Id})$ *is corepresented by $P_1(K^{\mathrm{cyc}})$.*

Proof. We can use the adjunction established in the previous point by theorem 3.2.11, thus giving an equivalence

$$\mathrm{Hom}_{\mathrm{TrThy}^{\mathrm{ex}}_{\Delta}}(P_1(K^{\mathrm{cyc}}), T) \simeq \mathrm{Nat}(K, T_{\mathrm{Id}}).$$

A priori, on the right hand side, we are considering K and T_{Id} as localizing invariants, but as far as the space of natural transformations is concerned, we might as well consider them as additive invariants. In this case, theorem 3.2.2 gives us a further equivalence

$$\mathrm{Nat}(K, T_{\mathrm{Id}}) \simeq \mathrm{Nat}(\widetilde{K}, \widetilde{T_{\mathrm{Id}}}).$$

Now combining theorem 3.2.4 and the Yoneda lemma [22, 5.1.5.2.] we get that this space of natural transformations can be understood as $\widetilde{T_{\mathrm{Id}}}(U_{\mathrm{add}}(\mathrm{Sp}^{\omega}))$. Now the result follows by definition of $\widetilde{-}$ as inverse to U^*_{add} and by definition of T_{Id}. ☐

3.4.3 Classifying trace theories and the Dundas-McCarthy theorem

The natural appearance of $ev_{(\mathrm{Sp},\mathrm{Id})}$ might lead us to considerations of the effect of this functor on trace theories, trying to make a further study of $P_1 K^{\mathrm{cyc}}$ as simple as could be, . For this, given a trace theory T, lets see what we can say about $T(\mathcal{C}_0, \vec{M})$. The fact that trace theories invert coCartesian edges in particular means that the map $T(\mathcal{C}_0; M_1, ..., M_0) \to T(\mathcal{C}_0; M_0 \circ ... \circ M_1)$, which is T of a coCartesian edge lying over a map $[n]_{\Lambda} \to [0]_{\Lambda}$ in Λ^{op}, is an equivalence. So we need only study $T(\mathcal{C}; M)$. Now $(M : \mathcal{C} \to \mathcal{C}) \in Pr^L_{st,(\omega)}$ is determined by its value on $\mathcal{C}^{\omega}$, in the sense that, denoting by よ $: \mathcal{C}^{\omega} \to \mathcal{C}$ the obvious inclusion, we have $M \simeq$ よ$_! M|_{\mathcal{C}^{\omega}}$. Now since M is a cocontinuous endofunctor of a stable category, it is exact, which we can interpret as saying that M is compatible with the Sp-enrichment of $\mathcal{C}$. Now, using the coend formula for enriched Kan extensions[10], we write よ$_! M|_{\mathcal{C}^{\omega}} \simeq \int_{x \in \mathcal{C}^{\omega}} \mathrm{hom}(x, -) \otimes M(x)$. What this says is that we can assume that M is a colimit of functors of the form $\mathrm{hom}(x, -) \otimes y$ with x a compact object of $\mathcal{C}$. So we only need to evaluate $T(\mathcal{C}, \mathrm{colim}_{x,y} \mathrm{hom}(x, -) \otimes y)$, but a priori T has no reason to interact in any agreeable way with the colimit appearing in this expression. The easy fix for this is the following definition, and we just need to hope we aren't overly restricting the class of trace theories we are considering.

Definition 3.4.36. ([30, 4.2.5.] and [30, 4.2.7.]) We call a trace theory $T : \Lambda_{st} \to \mathrm{Sp}$ cocontinuous if for any $\mathcal{C} \in Pr^L_{st,(\omega)}$ the functor $T(\mathcal{C}; -) : \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \to \mathrm{Sp}$ is cocontinuous. We indicate that we are considering the full subcategory of cocontinuous trace theories with an L as a superscript.

[10]The author was unable to find sources to justify that this argument can in fact be done ∞-categorically, but following a conversation with my supervisor Ching, it became clear this is the moral reason for the statement to hold. Thus, we decided to include the argument.

Remark 3.4.37. We are easily reassured that this is not too restrictive, since we know by design that P_1K^{cyc} is exact, so in order for the main trace theory we care about to be covered by the above result, we only need to show $P_1K^{\mathrm{cyc}}(\mathcal{C},-) : \mathrm{Fun}^L(\mathcal{C},\mathcal{C}) \to \mathrm{Sp}$ is finitary. We know from theorem 3.2.6 that K-theory is finitary, and from remark 3.4.20 we know K^{cyc} is finitary since colimits commute with colimits. And so the result follows from the fact that P_1 of something finitary is finitary.

With this, we see that if we are considering only cocontinuous trace theories, we are reduced to understanding $T(\mathcal{C}, \hom(x.-) \otimes y)$. We can write $\hom(x,-) \otimes y$ more explicitly as the composition $\mathcal{C} \xrightarrow{\hom(x,-)} \mathrm{Sp} \xrightarrow{-\otimes y} \mathcal{C}$. And so using cyclic invariance, which is the core property we designed trace theories for, we have $T(\mathcal{C}, \hom(x,-) \otimes y) \simeq T(\mathrm{Sp}, \hom(x, - \otimes y))$, which is cocontinuous since x is compact, and so we are restricted to understanding $T(\mathrm{Sp}, \hom(x,y) \otimes -)$. Now we use cocontinuity of T once more to reduce to $T(\mathrm{Sp}, \mathbb{S} \otimes -) \simeq T(\mathrm{Sp}, \mathrm{Id})$, since Sp is generated under colimits (which T preserves) by $\mathbb{S}$. In particular, a cocontinuous trace theory is fully determined by its evaluation at $(\mathrm{Sp}, \mathrm{Id})$, which we expect is free to be anything. This idea is made precise in the following result.

Theorem 3.4.38. *([30, 4.2.11.]) Evaluation at* $(\mathrm{Sp}, \mathrm{Id})$ *induces equivalences*

$$\mathrm{TrThy}^L_\Delta(\mathrm{Sp}) \to \mathrm{Sp}$$

and

$$\mathrm{TrThy}^L(\mathrm{Sp}) \to \mathrm{Sp}^{BS^1}.$$

This suggests an obvious strategy for explicitly determining P_1K^{cyc}, by evaluating $P_1K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{Id})$, and then finding a trace theory which evaluates to the same spectrum at $(\mathrm{Sp}, \mathrm{Id})$. In fact, since P_1K^{cyc} corepresents $ev_{(\mathrm{Sp},\mathrm{Id})}$, it is not too hard to compute $P_1K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{Id})$.

Corollary 3.4.39. *([30, 4.2.25.]) There is an equivalence*

$$P_1K^{\mathrm{cyc}}(\mathrm{Sp}, \mathrm{Id}) \simeq \mathbb{S}.$$

Proof. By the previous theorem, $ev_{(\mathrm{Sp},\mathrm{Id})} : \mathrm{TrThy}^L_\Delta(\mathrm{Sp}) \to \mathrm{Sp}$ is an equivalence, and so we can fix a left adjoint inverse $L : \mathrm{Sp} \to \mathrm{TrThy}^L_\Delta(\mathrm{Sp})$. By representability of $ev_{(\mathrm{Sp},\mathrm{Id})}$ we can identify $L(\mathbb{S})$ as P_1K^{cyc}, using the following chain of equivalences:

$$\mathrm{Hom}_{\mathrm{TrThy}^L_\Delta}(L(\mathbb{S}), -) \simeq \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, ev_{(\mathrm{Sp},\mathrm{Id})}(-))$$

$$\simeq \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, \mathrm{Hom}_{\mathrm{TrThy}^L_\Delta}(P_1K^{\mathrm{cyc}}, -)) \simeq \mathrm{Hom}_{\mathrm{TrThy}^L_\Delta}(P_1K^{\mathrm{cyc}}, -).$$

Now using that $ev_{(\mathrm{Sp},\mathrm{Id})}$ and L are mutually inverse we get that $ev_{(\mathrm{Sp},\mathrm{Id})}(P_1K^{\mathrm{cyc}}) \simeq ev_{(\mathrm{Sp},\mathrm{Id})}(L(\mathbb{S})) \simeq \mathbb{S}$, which is what we wanted to show. □

The question remains, does the strategy suggested above actually lead to any interesting identification of P_1K^{cyc}? For this we need to introduce topological Hochschild homology (THH), a computationally nice invariant which extends classical Hochschild homology from $\mathbb{Z}$-algebras (see [37, 9.]) all the way to small stable categories. Our introduction will be brief, completely anachronistic and will be based on [21], which we encourage the interested reader to consult.
The hope in understanding P_1K^{cyc}, is that since K is universal, and so in a sense a "fundamental" object, we might expect P_1K^{cyc} to be quite basic as a trace theory. The name "trace theory" was motivated by the classical trace function of linear algebra, but since there is a definition of trace which holds for any endomorphism of any dualizable object in any monoidal category, one might wonder whether the connection between trace theories and the linear algebra trace runs deeper. One way this could happen is, since trace theories are determined by their value on the fiber over $[0]_\Lambda$ of $\Lambda_{st} \to \Lambda^{\mathrm{op}}$, whose objects are a choice a compactly generated category $\mathcal{C}$ and a cocontinuous endomorphism $M : \mathcal{C} \to \mathcal{C}$, is if every $\mathcal{C}$ in consideration happens to be dualizable, so that we have a well defined trace of M. This happens by the following result.

Proposition 3.4.40. *(The case* $\mathcal{E} = \mathrm{Sp}$ *of [21, 4.10.]) Every stable compactly generated category* $\mathcal{C}$ *is dualizable in the category* $Pr^L_{st,(\omega)}$.

Therefore, we can define $tr(\mathcal{C}, M)$ on every pair $(\mathcal{C}, M : \mathcal{C} \to \mathcal{C})$ we would want in order to define a trace theory, and furthermore $tr(\mathcal{C}, M)$ can be seen as a spectrum because the trace of an endomorphism of a dualizable object is by construction an endomorphism of the appropriate monoidal unit, which in this case means $tr(\mathcal{C}, M) \in \mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$. It turns out this construction can be made functorial.

Proposition 3.4.41. *([21, 2.9.]) There is a functor $Tr : \mathrm{End}(Pr^L_{st,(\omega)}) \to \mathrm{Sp}$, where $\mathrm{End}(Pr^L_{st,(\omega)})$ is equivalent to the fiber over $[0]_\Lambda$ of Λ_{st}, which sends a pair $(\mathcal{C}, M)$ to the trace of M.*

But we want more than that, we want it to be a trace theory, with all of the extra structure that entails. This is the content of [30, 4.2.2.]. And this in fact gives a suitable definition of THH, which is furthermore equivalent to the more classical definition of THH (see for example [25]) as proven in [21, 4.5.]. Since the trace $Tr(f)$ of an endomorphism of the unit $f : \mathbf{1} \to \mathbf{1}$ is f itself, we get the following result.

Corollary 3.4.42. *We have $THH(\mathrm{Sp}, \mathrm{Id}) \simeq \mathbb{S}$, in particular we have an isomorphism of trace theories $THH \simeq P_1 K^{\mathrm{cyc}}$.*

Proof. First, we observe that THH is exact in the bimodule variable by [16, 4.25.] and it is finitary in the bimodule variable as it can be realized as a colimit (see [16, 4.6.]). In particular, $THH(\mathcal{C}, -)$ is cocontinuous, so theorem 3.4.38 applies. By the above paragraph and the fact that THH is the trace functor, we have $THH(\mathrm{Sp}, \mathrm{Id})$ must correspond to the identity of Sp, which is $\mathbb{S}$ under the identification $\mathrm{Fun}^L(\mathrm{Sp}, \mathrm{Sp}) \simeq \mathrm{Sp}$ which we use. The result then follows from a direct application of theorem 3.4.38 and corollary 3.4.39. ☐

At this point, we have an equivalence $\delta : P_1 K^{\mathrm{cyc}} \to THH$, the best we could hope for is to identify this map. By corollary 3.4.34, this map corresponds to a map of localizing invariants $K \to THH_{\mathrm{Id}}$. The space of natural maps between two localizing invariants is the same as if we consider them as additive invariants, which allows us to deduce.

Proposition 3.4.43. *([4, 10.4.]) The space of natural transformation $\mathrm{Nat}(K, THH_{\mathrm{Id}})$ as additive invariants is $\mathbb{S}$. In particular, the homotopy class of a natural map $K \to THH_{\mathrm{Id}}$ corresponds to an integer via the isomorphism $\pi_0(\mathbb{S}) \simeq \mathbb{Z}$.*

Proof. By theorem 3.2.2, we have an equivalence $\mathrm{Nat}(K, THH_{\mathrm{Id}}) \simeq \mathrm{Nat}(\widetilde{K}, \widetilde{THH_{\mathrm{Id}}})$. Now $\widetilde{K}$ is corepresentable by theorem 3.2.4, so this space is equivalent, by the Yoneda lemma, to $\widetilde{THH_{\mathrm{Id}}}(U_{\mathrm{add}}(\mathrm{Sp}^\omega))$, which by chasing definitions is $THH(\mathrm{Sp}, \mathrm{Id})$, which we discussed is equivalent to $\mathbb{S}$ above. ☐

We would like to show that the homotopy class of the equivalence δ corresponds to the integer 1. The reason for this is that the natural map corresponding to 1 under the equivalence described in the above result has been shown in [4, 10.6.] to be the Dennis trace map, which is a famous map $K \to THH_{\mathrm{Id}}$.

Since f is an equivalence if and only if $-f$ is one, we can replace δ by $-\delta$ as the equivalence $K \to THH_{\mathrm{Id}}$ we are considering. Thus we are reduced to proving the homotopy class of δ corresponds to ± 1, which amounts precisely to showing δ is not divisible. We proceed by contradiction. If δ were divisible say by n, that is to say we have an equivalence $\delta \simeq n \cdot \delta'$, then also

$$\mathrm{Id}_{THH} \simeq \delta \circ \delta^{-1} \simeq n \cdot \delta' \circ \delta^{-1} \simeq n(\delta' \circ \delta^{-1}).$$

What this shows is that divisibility of δ implies divisibility of the identity of THH_{Id}. Evaluating at $(\mathrm{Sp}, \mathrm{Id})$, this would imply that

$$\mathrm{Id} : \mathbb{S} \simeq THH(\mathrm{Sp}, \mathrm{Id}) \to THH(\mathrm{Sp}, \mathrm{Id}) \simeq \mathbb{S}$$

is also divisible. This is the desired contradiction and so we get the following result, known as the Dundas-McCarthy theorem.

Theorem 3.4.44. *([30, 4.2.1.]) The Dennis trace map* $K \to \mathrm{THH}$ *induces an equivalence of trace theories*

$$P_1 K^{\mathrm{cyc}} \simeq \mathrm{THH}\,.$$

Remark 3.4.45. Since this was a point of confusion for the author at the beginning of this project, we mention that the above theorem is NOT the Dundas-McCarthy-Goodwillie theorem, which is a different, but very much related theorem. For an insight into the Dundas-McCarthy-Goodwillie theorem, and its relation to the above result, we point the reader to [31] which proves the theorem in a modern language.

4 The comonad $\Sigma^\infty\Omega^\infty$ for $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$

4.1 Motivation of the result

We start with an objectwise understanding of the composition $L_{\mathcal{C}}(\mathrm{Lace}(\mathcal{C}, M))$ (i.e. $\Sigma^\infty\Omega^\infty$ by proposition 3.3.9), which will motivate our further investigation in the next subsections. To this end, it will be easier to ponder the interaction of L and Lace if they both consider bimodules in the same way. In particular, we seek to understand $L(f)$ as a bimodule in the sense of a cocontinuous functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.

Proposition 4.1.1. *Translating from perspective (i) in 3.3.2 to perspective (iii), the functor $L_{\mathcal{C}}$ is given by $(f : \mathcal{D} \to \mathcal{C}) \mapsto \mathrm{Ind}(f) \circ f_!(よ_{\mathcal{D}})$. Furthermore, translating to perspective (iv), we can then describe $L(f)$ as $\mathrm{Ind}(f)\,\mathrm{Ind}(f)^R$ where $\mathrm{Ind}(f)^R$ is a right adjoint of $\mathrm{Ind}(f)$.*

Proof. Consider a map $f : \mathcal{D} \to \mathcal{C}$ as an object of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ so that in particular f is exact. Thus we may consider $L_{\mathcal{C}}(f)$, written as $L(f)$ for the remainder of the proof, which can be expressed as a left Kan extension in the following diagram

$$\begin{tikzcd} \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \arrow[r, "\mathrm{Hom}_{\mathcal{D}}"] \arrow[d, "f^{\mathrm{op}}\times f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathrm{Sp} \arrow[d, "\mathrm{Id}"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[r, "L(f)"'] & \mathrm{Sp}\,. \end{tikzcd}$$

Recall that Ind sends a category $\mathcal{C}$ to the subcategory of finite limit preserving functors of $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}, \mathcal{S})$ (combine [22, 5.3.2.1.], [22, 5.3.2.3.][11],[22, 5.3.5.1.] and [22, 5.3.5.4.][12]). By [23, 1.4.2.23.] $\mathcal{D}$ (resp. $\mathcal{C}$) being stable, we can replace $\mathrm{Ind}(\mathcal{D})$ (resp. $\mathrm{Ind}(\mathcal{C})$) by the category of left exact functors from $\mathcal{D}$ (resp. $\mathcal{C}$) to Sp. By [23, 1.1.4.1.] this is immediately equivalent to the category of right exact functors from $\mathcal{D}$ (resp. $\mathcal{C}$) to Sp.
Now we may curry[13] the above diagram in $\mathrm{Cat}^{\mathrm{ex}}$ in order to get the diagram

$$\begin{tikzcd} \mathcal{D} \arrow[r, "よ_{\mathcal{D}}"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & \mathrm{Ind}(\mathcal{D}) \\ \mathcal{C} \arrow[r, "L(f)"'] & \mathrm{Ind}(\mathcal{C}). \arrow[u, "f^*"'] \end{tikzcd}$$

We have allowed and will continue to allow ourselves the mild abuse of notation of not modifying the notation for α and $L(f)$, even though technically these are the corresponding curried maps. Notice that the pair $(L(f), \alpha)$ still has a universal property. Specifically, $(L(f), \alpha)$ is initial among those functors $F : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ with a natural transformation $\alpha_F : よ_{\mathcal{D}} \to f^* \circ F \circ f$. Now we can use the adjunction for left Kan extensions $f_! \vdash - \circ f$, followed by the adjunction $f^* - \circ \vdash \mathrm{Ind}(f) \circ -$, which is a direct consequence of [22, 5.3.5.13.], to rephrase the universal property of $L(f)$ as being the initial functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ with a natural transformation from $\mathrm{Ind}(f) \circ f_!(よ_{\mathcal{D}})$. But surely the initial such functor is $\mathrm{Ind}(f) \circ f_!(よ_{\mathcal{D}})$ itself.

We now prove the second part of our statement. Observe the following, slightly cumbersome, diagram where all of the back to front arrows are corestricted Yoneda embeddings, which neatly summarizes what happens when applying the Ind functor to the diagram defining $L(f)$ in perspective

[11] The dual of these two statements combine to give our working definition of a left exact functor as one that preserve finite limits.

[12] These two results give the description of $\mathrm{Ind}(\mathcal{C})$ as the full subcategory of $\mathcal{P}(\mathcal{C})$ on finite limit preserving functors. This and the previous footnote explain how the four results from [22] combine into the statement we claimed.

[13] If we wish to be pedantic, this is done with an intermediate step of replacing $\mathcal{D}^{\mathrm{op}} \times \mathcal{D}$ by $\mathcal{D}^{\mathrm{op}} \otimes \mathcal{D}$ (and similarly for $\mathcal{C}$), which we can do given the exactness of the functors in consideration.

(iii)

$$\begin{tikzcd}
\mathcal{D} \arrow[rr, "\text{よ}_{\mathcal{D}}"] \arrow[d, "f"'] \arrow[ddr] & & \mathrm{Ind}(\mathcal{D}) \arrow[d, "\mathrm{Ind}(f)"] \arrow[ddr] & \\
\mathcal{C} \arrow[rr, "L(f)"'] \arrow[urr, "f_!(\text{よ}_{\mathcal{D}})"] \arrow[ddr] & & \mathrm{Ind}(\mathcal{C}) \arrow[ddr] & \\
& \mathrm{Ind}(\mathcal{D}) \arrow[rr, "\mathrm{Id}_{\mathcal{D}}"] \arrow[d, "\mathrm{Ind}(f)"] & & \mathrm{Ind}(\mathcal{D}) \arrow[d, "\mathrm{Ind}(f)"] \\
& \mathrm{Ind}(\mathcal{C}) \arrow[rr, "\mathrm{Ind}(L(f))"'] \arrow[urr, "\mathrm{Ind}(f_!(\text{よ}_{\mathcal{D}})"] & & \mathrm{Ind}(\mathcal{C}).
\end{tikzcd}$$

We want to understand the front diagonal map better. By [22, 5.3.5.10.] restriction along a Yoneda embedding induces an equivalence

$$\mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

Therefore, we deduce by comparison of universal properties that $\mathrm{Ind}(f_!(\text{よ}_{\mathcal{D}}))$ has the universal property of the Kan extension with respect to those functors which preserves colimits. We can therefore see it must in fact be equivalent to $\mathrm{Ind}(f)_!(\mathrm{Id}_{\mathcal{D}})$. And as a Kan extension it has a universal property with respect to all functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{D})$. Which since $\mathrm{Ind}(f)_!(\mathrm{Id}_{\mathcal{D}})$ is cocontinuous, gives $\mathrm{Ind}(f)_!(\mathrm{Id}_{\mathcal{D}})$ a universal property in the category of cocontinuous functors $\mathcal{C} \to \mathrm{Ind}(\mathcal{D})$. But left Kan extension of the identity gives the right adjoint of the map along which we Kan extended, i.e. $\mathrm{Ind}(f)^R$, thus concluding the proof. $\square$

In particular, the above shows that $L(f)$ can be seen as a colimit preserving comonad. In particular, we expect the comonad $L_{\mathcal{C}} \circ \mathrm{Lace}(\mathcal{C}, -) : \mathrm{BiMod}(\mathcal{C}) \to \mathrm{BiMod}(\mathcal{C})$ to send an arbitrary cocontinuous functor $M : \mathcal{C} \to \mathcal{C}$ to a cocontinuous comonad on $\mathcal{C}$. The most natural guess for the structure of $L_{\mathcal{C}} \circ \mathrm{Lace}(\mathcal{C}, -)$ is then the title of the next subsection.

4.2 The cofree finitary comonad

We want to show that the monoidal category $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ admits cofree coAlgebras whenever $\mathcal{C}$ is stable and presentable, in particular when $\mathcal{C}$ is Ind-completed from an object of $\mathrm{Cat}^{\mathrm{ex}}$. We will denote this category of coAlgebras simply by $\mathrm{coMonad}^L(\mathcal{C})$. To do this, notice that the composition monoidal structure can be recorded as a coCartesian fibration of ∞-operads $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})^{\otimes} \to \mathbf{Assoc}^{\otimes}$ (see [23, 4.1.1.10.]). We will proceed following section 3.1 of [24], where Lurie explains when monoidal ∞-categories admit cofree cocommutative coAlgebras, but will need to slightly modify the proof since we do not have any commutativity assumptions. The first tool is the following result from higher algebra which, using the identification $\mathrm{coAlg}(\mathcal{D}) := \mathrm{Alg}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}}$, readily shows that $\mathrm{coMonad}^L(\mathcal{C})$ admits all colimits and that the forgetful functor to $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ preserves them.

Proposition 4.2.1. *([23, 3.2.2.5.]) Let $p : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a coCartesian fibration of ∞-operads, K a simplicial set such that for all $X \in \mathcal{O}^{\otimes}$ the fiber $\mathcal{C}_X$ admits K-indexed limits. Then the ∞-category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admits K-indexed limits and the evaluation at X functor preserves these limits.*

This implies that $\mathrm{Alg}(\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})^{\mathrm{op}})$ admits all the limits which are present in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})^{\mathrm{op}}$ because the associative operad is a one object operad, and the only "evaluation functor" for associative algebras in an ∞-category is identified with the forgetful functor. Taking opposite categories, this means that $\mathrm{coMonad}^L(\mathcal{C})$ admits all colimits present in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ and that the forgetful functor $\mathrm{coMonad}^L(\mathcal{C}) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ preserves these.

Showing that the category of cocontinuous endofunctors of $\mathcal{C}$ admits cofree coAlgebras is to show, in our case via the adjoint functor theorem, that the forgetful functor $U : \mathrm{coMonad}^L(\mathcal{C}) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is a left adjoint. In order to do this, we first notice that $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is presentable by [22, 5.5.3.8] and therefore by the above paragraph it follows that $\mathrm{coMonad}^L(\mathcal{C})$ admits all small colimits.

All that remains to show is that the category $\mathrm{coMonad}^L(\mathcal{C})$ is accessible, as then the adjoint functor theorem allows us to conclude. First we will notice that the monoidal structure on $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is accessible. To do this we need the following result.

Proposition 4.2.2. *[20, D.4.11.] Let $\mathcal{C}, \mathcal{D}$ and I be categories and let $(f_i)_I$ be a diagram in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, such that each f_i admits a right adjoint g_i. Suppose that either the colimit of the f_i over I or the limit over I^{op} of the g_i exists. Then we have an adjunction*

$$\mathrm{colim}_I\, f_i : \mathcal{C} \rightleftarrows \mathcal{D} : \lim_{I^{\mathrm{op}}} g_i.$$

In particular, colimits in $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D})$ can be computed in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proposition 4.2.3. *The composition functor $- \circ - : \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \times \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is cocontinuous separately in each variable.*

Proof. Fix a cocontinuous functor $F : \mathcal{C} \to \mathcal{C}$. We will show that $F \circ -$ is cocontinuous when seen as an endofunctor of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ which will allow us to conclude as colimits of left adjoints can be computed in this latter category. The functor $F \circ -$ is itself a left adjoint with right adjoint $F^R \circ -$ where F^R is the right adjoint of F. This implies that $F \circ -$ preserves colimits, thus concluding the first half of this result. Now we need to show that $- \circ F$ preserves colimits. Because colimits of left adjoints can be computed in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ they can in particular be computed objectwise. From this, it follows that $- \circ F$ preserves colimits. ☐

Before showing with this that the category of coAlgebras is accessible, we recall the following result.

Proposition 4.2.4. *(Specialization of [22, 5.4.7.11.] by direct application of [22, 5.4.7.12.] and [22, 5.4.7.13.]) Let Acc be the subcategory of Cat_∞ (not necessarily small ∞-categories) whose objects are accessible ∞-categories, and the morphisms are the accessible functors. Let $p : X \to S$ be a map of simplicial sets with S small which is a (co)Cartesian fibration classified by a functor $S \to \mathrm{Acc} \subset \mathrm{Cat}_\infty$. Let $\mathcal{E} \subset S$ be a set of edges and let Y be the full subcategory of the category of sections of p, denoted by $\mathrm{Map}_p(S, X)$, of those sections which send edges $e \in \mathcal{E}$ to (co)Cartesian edges.*
Then Y is accessible and for any $\mathcal{C} \in \mathrm{Acc}$ a functor $F : Z \to Y$ is accessible if and only if for every vertex in S the composite map $Z \to Y \to X_s$ is accessible.

The following result is a slight generalization of [24, 3.1.3.]

Proposition 4.2.5. *Let $\mathcal{C}$ be a monoidal ∞-category. Suppose that $\mathcal{C}$ is accessible and that the tensor product functor is accessible. Then the ∞-category $\mathrm{coAlg}(\mathcal{C})$ of coAlgebras in $\mathcal{C}$ is accessible.*

Proof. We encode the monoidal structure of $\mathcal{C}$ as a coCartesian fibration of ∞-operads $q : \mathcal{C}^{\otimes} \to \mathbf{Assoc}^{\otimes}$. And so associative algebras become exactly sections of q which carry inert morphisms to inert morphisms. We consider the opposite monoidal structure, classified by $p : (\mathcal{C}^{\mathrm{op}})^{\otimes} \to \mathbf{Assoc}^{\otimes}$, whose category of associative algebras is the opposite category of the category of interest to us.

So the category we are interested in is the subcategory $Y \subset \mathrm{Fun}_p(\mathbf{Assoc}^{\otimes}, (\mathcal{C}^{\mathrm{op}})^{\otimes})^{\mathrm{op}}$ spanned by those functors which carry inert edges to inert edges. Given that we are considering a category of sections, it is direct that the image of an inert edge by any functor lies over an inert edge. So the missing hypothesis for a functor belonging to Y becomes that we require inert edges be sent to coCartesian edges.

Therefore, it suffices to verify that Y is considered by proposition 4.2.4. In order to find oneself in the appropriate setting we rewrite $\mathcal{D}$ as $\mathrm{Fun}_{p^{\mathrm{op}}}((\mathbf{Assoc}^{\otimes})^{\mathrm{op}}, \mathcal{C}^{\otimes})$. Under this identification, Y becomes the full subcategory on functors sending inert edges to Cartesian edges.

The fact that p is a coCartesian fibration of ∞-operads implies that p^{op} is a Cartesian fibration. Furthermore the accessibility assumption on both $\mathcal{C}$ and the tensor product functor say exactly that p^{op} is classified by a functor $\mathbf{Assoc}^{\otimes} \to \mathrm{Acc}$. Therefore, taking $\mathcal{E}$ to be the set of inert edges leads proposition 4.2.4 to imply precisely that Y is accessible, which is what we wanted to show.

□

From all of the above we deduce the desired result.

Corollary 4.2.6. *Let $\mathcal{C}$ be a stably presentable category. Then:*

(i) The category $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is presentable and therefore accessible.

(ii) This category is monoidal via functor composition and the tensor product is accessible.

(iii) The category of finitary comonads on $\mathcal{C}$ is therefore accessible.

(iv) We have an adjunction $U : \mathrm{coMonad}^L(\mathcal{C}) \rightleftarrows \mathrm{Fun}^L(\mathcal{C}, \mathcal{C}) : \mathrm{coFree}$.

4.3 Laced objects are coAlgebras for a cofree comonad

In this section, we will sketch a proof identifying the comonad $L_{\mathcal{C}} \circ \mathrm{Lace}(\mathcal{C}, -)$ with the "cofree comonad comonad". We will as clearly as possible point out the steps for which the technical language needed to formulate and precisely prove certain results eluded the author, and will as much as possible point to literature which might help in obtaining the necessary mathematical baggage. The first such point, concerning various possible meanings of the comonad functor $\mathrm{coEnd} : \mathrm{Cat}^L_{/\mathcal{C}} \to \mathrm{coMonad}(\mathcal{C})$, is relegated to appendix A where we undertake a relatively thorough investigation of what we know, what we need, and what remains to be precisely proven.
In particular, there is the coEnd functor which is proven to be left adjoint to the the category of coAlgebras functor in [19] and the functor which appears in the adjunction of lemma A.3.5, which we need to be the same, but at no point do we explicitly prove this. In the same appendix, we also do our best to compare this coEnd construction to the one in [3] which already made an appearance when we were discussing Koszul duality.

Remark 4.3.1. Our main perspective on the comonad functor comes from [19]. However, the key theorem which we needed of Heine's paper (our theorem A.2.3) was also proven, in a slightly different setting, by Haugseng in [17, 5.8.]. The author originally leaned more to the first of these two papers, since the second one requires some yet to be proven results about the Gray tensor product and because it involves the language of ∞-cosmoi, which the author had never heard of until that point. However, it is possible that the perspective in [17] is more suited to our purposes. For the reader who wishes to explore this avenue further, we believe a good place to understand the context of [17] is [32], the second paper of Riehl and Verity understanding the category of ∞-categories as a specific example of an ∞-cosmos.

The next point of order is that in so far as $\mathrm{Lace}(\mathcal{C}, -)$ can be seen as a lax limit, as we pointed out in remark 3.3.12, we expect maps into $\mathrm{Lace}(\mathcal{C}, M)$ to correspond to maps to $g : \mathcal{D} \to \mathcal{C}$ which make $よ, M : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ "lax equal" (in a prescribed manner) upon precomposition. In other words, we expect maps to $\mathrm{Lace}(\mathcal{C}, M)$, to be given by the data of a map $g : \mathcal{D} \to \mathcal{C}$ and a natural map $よ \circ g \to M \circ g$. Since we consider $\mathrm{Lace}(\mathcal{C}, -)$ as landing in $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ rather than just in $\mathrm{Cat}^{\mathrm{ex}}$, the expected universal property is the following equivalence:

$$\mathrm{Fun}^{\mathrm{ex}}_{/\mathcal{C}}(g : \mathcal{D} \to \mathcal{C}, \gamma_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}) \simeq \mathrm{Nat}(\mathrm{Ind}(g), M \circ \mathrm{Ind}(g)),$$

where the appearance of $\mathrm{Ind}(g)$ over $よ \circ g$ follows from the properties of $\mathrm{Ind}(-)$, $よ$ and the relation between these. Indeed, using the equivalence of [22, 5.3.5.10.] provided by precomposition by $よ$, we have $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Ind}(\mathcal{C}))$, which sends $よ \circ g$ to $\mathrm{Ind}(g)$ by the text preceding [22, 5.3.5.13.]. To replace $M \circ g$ by $M \circ \mathrm{Ind}(g)$, we use the same ideas, only indulging in some minor abuse in identifying $M : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ with $\mathrm{Ind}(M) : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Ind}(\mathcal{C})) \simeq \mathrm{Ind}(\mathcal{C})$, justified by the equivalent perspectives of definition 3.3.2 and needing to recall $\mathrm{Ind}(M \circ g) \simeq \mathrm{Ind}(M) \circ \mathrm{Ind}(g)$, as is

easily verified using the equivalence $\mathrm{Fun}(\mathrm{Ind}(\mathcal{C}),\mathrm{Ind}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C},\mathrm{Ind}(\mathcal{C}))$.
Objectwise, the above equivalence in fact follows directly from our definition of $\mathrm{Lace}(\mathcal{C}, M)$ as the pullback

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C}, M) \arrow[r] \arrow[d] & \mathcal{C}^{\Delta^1} \arrow[d, "(\mathrm{Dom},\mathrm{Cod})"] \\
\mathcal{C} \arrow[r, "(よ, M)"'] & \mathcal{C} \times \mathcal{C}.
\end{tikzcd}$$

Notice that the LHS of the equivalence can be seen as a functor $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \times \mathrm{Bimod}(\mathcal{C}) \to \mathcal{S}$, by realizing it as the composition

$$\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \times \mathrm{Bimod}(\mathcal{C}) \xrightarrow{(\mathrm{Id},\mathrm{Lace}(\mathcal{C},-))} \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \times \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \xrightarrow{\mathrm{Fun}^{\mathrm{ex}}_{/\mathcal{C}}} \mathcal{S}.$$

Notice also that the RHS of the equivalence can also be seen as a functor, this time by direct application of lemma A.3.2. So in order to make the universal property of $\gamma_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ in $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ precise, we need to make the equivalence above a natural one, with the LHS and RHS functors in the way we just described.

The problem with obtaining a precise proof of this statement, is that the author's understanding of $\mathrm{Lace}(\mathcal{C}, -)$ as a lax limit is too imprecise in order to compare $\mathrm{Fun}(- : - \to \mathcal{C}, \gamma_\bullet\, \mathrm{Lace}(\mathcal{C}, \bullet) \to \mathcal{C})$ to the precise model of $\mathrm{Nat}(\mathrm{Ind}(-), \bullet \circ \mathrm{Ind}(-))$ we constructed in lemma A.3.2. We record this desired result in the following proposition.

Conjecture 4.3.2. *There is a natural equivalence*

$$\mathrm{Fun}^{\mathrm{ex}}_{/\mathcal{C}}(g : \mathcal{D} \to \mathcal{C}, \gamma_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}) \simeq \mathrm{Nat}(\mathrm{Ind}(g), M \circ \mathrm{Ind}(g)),$$

between the functors as described above.

Remark 4.3.3. In order to precisely prove the above statement, I would need to develop further comfort with the fibrational language which is at the heart of most strategies to deal with all the higher coherences required to apprehend higher categorical structures, which as I understand is a skill which comes only with practice. Furthermore, an understanding of lax limits, their universal properties and how to phrase them fibrationally is probably essential to proving the above result. For the reader who might want to pursue this, we believe [12] to be a good place to start these investigations.

Combining the above conjecture with lemma A.3.5 yields the following result.

Theorem 4.3.4. *There is an adjunction* $\mathrm{coEnd}(\mathrm{Ind}(-)) \dashv \mathrm{Lace}(\mathcal{C}, -)$.

Proof. Composing the natural equivalence $\alpha * \mathrm{Ind}(-) : \mathrm{Nat}(\mathrm{coEnd}(\mathrm{Ind}(g)), M) \to \mathrm{Nat}(\mathrm{Ind}(g), M \circ \mathrm{Ind}(g))$ with the natural equivalence $\mathrm{Nat}(\mathrm{Ind}(g), M \circ \mathrm{Ind}(g)) \to \mathrm{Fun}^{\mathrm{ex}}_{/\mathcal{C}}(g : \mathcal{D} \to \mathcal{C}, \gamma_M : \mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C})$ exactly gives the mapping space characterization of the statement that $\mathrm{coEnd}(\mathrm{Ind}(-))$ is left adjoint to $\mathrm{Lace}(\mathcal{C}, -)$. □

From the dissection of the left adjoint of $\mathrm{Lace}(\mathcal{C}, -)$, we would like to infer a similar factorization of $\mathrm{Lace}(\mathcal{C}, -)$. However, we have not quite expressed the left adjoint as a composite of adjoints. Indeed, making everything explicit, the left adjoint is the corestriction to cocontinuous endofunctors of the following composition

$$\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \xrightarrow{\mathrm{Ind}} \mathrm{Pr}^{str,L}_{st,/\,\mathrm{Ind}(\mathcal{C})} \hookrightarrow \mathrm{Cat}_{/\,\mathrm{Ind}(\mathcal{C})} \xrightarrow{\mathrm{coEnd}} \mathrm{coMonad}(\mathrm{Ind}(\mathcal{C})) \xrightarrow{U} \mathrm{Fun}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})).$$

The second arrow and the corestriction are the remaining obstruction to understanding $\mathrm{Lace}(\mathcal{C}, -)$ as a composition of adjoints. One way to conclude is if we can suitably restrict and corestrict the $\mathrm{coEnd} \dashv \mathrm{coAlg}$ to $\mathrm{Pr}^{str,L}_{st,/\,\mathrm{Ind}(\mathcal{C})}$ and $\mathrm{coMonad}^L(\mathrm{Ind}(\mathcal{C}))$. Once we prove this, the composition $\mathrm{coEnd}(\mathrm{Ind}(-))$ as it appears in the above theorem is exactly

$$\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \xrightarrow{\mathrm{Ind}} \mathrm{Pr}^{str,L}_{st,/\,\mathrm{Ind}(\mathcal{C})} \xrightarrow{\mathrm{coEend}} \mathrm{coMonad}^L(\mathrm{Ind}(\mathcal{C})) \xrightarrow{U} \mathrm{Fun}^L(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(cC)).$$

From this one deduces.

Theorem 4.3.5. *We have* $\mathrm{coAlg}(\mathrm{coFree}(-))^\omega \simeq \mathrm{Lace}(\mathcal{C}, -)$*. In particular the comonad associated to the right adjoint* $\mathrm{Lace}(\mathcal{C}, -)$ *sends a bimodule* M *to the bimodule underlying the cofree comonad* $\mathrm{coFree}(M)$.

Proof. We know that the left adjoint of $\mathrm{Lace}(\mathcal{C}, -)$ is $\mathrm{coEnd} \circ \mathrm{Ind} \circ U$. The right adjoint of coEnd is coAlg by (the (co)restricted version of) theorem A.2.3, the right adjoint of Ind is the functor which sends a category to its category of compact objects (in fact these are inverse equivalences of one another, see [22, 5.5.7.10.]) and the right adjoint of $U : \mathrm{coMonad}^L(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ is the cofree comonad of corollary 4.2.6. Now, by uniqueness of adjoints, we must have $\mathrm{Lace}(\mathcal{C}, -) \simeq \mathrm{coAlg}(\mathrm{coFree}(-))^\omega$. And the second part of our theorem follows at once. □

So all that remains is to prove the following result. We are not able to do this in full, and provide explanations for further pursuits in the proceeding remark.

Conjecture 4.3.6. *When* $\mathcal{C}$ *is stable presentable, the adjunction* $\mathrm{coEnd} : \mathrm{Cat}^L_{/\mathrm{Ind}(\mathcal{C})} \leftrightarrows \mathrm{coMonad}(\mathrm{Ind}(\mathcal{C})) :$ coAlg *restricts to an adjunction between* $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})}$ *and* $\mathrm{coMonad}^L(\mathrm{Ind}(\mathcal{C}))$.

We are able to prove most of this result, by using the following result.

Proposition 4.3.7. *([23, 4.2.3.4.]) Let* $\mathcal{A} \subset \widehat{\mathrm{Cat}}$ *be a subcategory of the category of (not necessarily small)* ∞*-categories which has the following properties :*

(i) The category $\mathcal{A}$ *admits all small limits and the inclusion* $\mathcal{A} \to \widehat{\mathrm{Cat}}$ *preserves finite limits.*

(ii) If X *belongs to* $\mathcal{A}$ *then* $\mathrm{Fun}(\Delta^1, X)$ *belongs to* $\mathcal{A}$.

(iii) If X *and* Y *belong to* $\mathcal{A}$*, a morphism* $\alpha : X \to \mathrm{Fun}(\Delta^1, Y)$ *belongs to* $\mathcal{A}$ *if and only if* $ev_0 \circ \alpha, ev_1 \circ \alpha : X \to Y$ *both belong to* $\mathcal{A}$.

Let $\mathcal{C}$ *be a monoidal* ∞*-category,* A *an algebra object of* $\mathcal{C}$ *and* $\mathcal{M}$ *an object of* $\mathcal{A}$*, left tensored over* $\mathcal{C}$*, such that* $A \otimes - : \mathcal{M} \to \mathcal{M}$ *is a morphism in* $\mathcal{A}$*. Then*

(i) The category $\mathrm{LMod}_A(\mathcal{M})$ *is an object of* $\mathcal{A}$.

(ii) For any $\mathcal{N} \in \mathcal{A}$*, a functor* $\mathcal{N} \to \mathrm{LMod}_A(\mathcal{M})$ *is in* $\mathcal{A}$ *if and only if the post-composition* $\mathcal{N} \to \mathcal{M}$ *by the forgetful functor* $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ *is in* $\mathcal{A}$.

(iii) The forgetful functor $\mathrm{LMod}_A(\mathcal{M}) \to \mathcal{M}$ *is in* $\mathcal{A}$.

Partial proof of proposition 4.3.6. Since it is obvious that the comonad associated to a strongly cocontinuous left adjoint is cocontinuous, it is clear how to lift to cocontinuous comonads the restriction of CoEnd to $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})}$. The more subtle point is, if we restrict coEnd to the category of cocontinuous comonads, how do we corestrict coAlg to $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})}$.
First notice that the inclusion $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})} \to \mathrm{Cat}_{/\mathrm{Ind}(\mathcal{C})}$ is "full on 2-cells and above", in the sense that for any higher cell of $\mathrm{Cat}_{/\mathrm{Ind}(\mathcal{C})}$ whose 1- and 0-faces lie in $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})}$, that higher cell automatically belongs to $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})}$. And so to corestrict coAlg, it suffices to study what happens on objects and morphism, i.e. on 0- and 1-cells.
We write $\mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C}))$ as $\mathrm{LMod}_Q(\mathrm{Ind}(\mathcal{C})^{\mathrm{op}})^{\mathrm{op}}$. This allows us to study properties of $\mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C}))$ by the proposition we recalled above. For stability, notice a category is stable if and only if the opposite category is stable. So to show $\mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C}))$ is stable, it suffices to show that the category of stable categories (and cocontinuous functors between them) satisfies the assumptions of the previous proposition. Condition (i) is [8, 6.1.1.], condition (ii) is [23, 1.1.3.1.] and condition (iii) is by pointwise computation of colimits in functor categories. And so, $\mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C}))$ is stable and the forgetful functor to $\mathrm{Ind}(\mathcal{C})$ is cocontinuous because $Q : \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{C})$ is cocontinuous.
We now discuss presentability. For this we consider the property of copresentability, which is said of a category $\mathcal{D}$ such that $\mathcal{D}^{\mathrm{op}}$ is presentable. We will show that the subcategory of copresentable categories and right adjoint satisfies the assumptions of the previous proposition, so that $\mathrm{LcoMod}_Q(\mathrm{Ind}(\mathcal{C})^{\mathrm{op}})$ is

copresentable, since $\mathcal{C}$ is presentable, which will imply the desired presentability. Since $(-)^{\mathrm{op}}$ is an equivalence, it can be made to preserve limits, so that condition (i) for copresentability will follow from condition (i) for presentable categories and left adjoints, which is [22, 5.5.3.13.]. Condition (ii) is [22, 5.5.3.6.] and the rest of the argument follows just as it did for stability.
Now we need to show that the map $U : \mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C})) \to \mathrm{Ind}(\mathcal{C})$ is strongly cocontinuous, and that for a map $\alpha : Q \to R$, the induced map $\mathrm{coAlg}(\alpha) : \mathrm{coAlg}_Q(\mathrm{Ind}(\mathcal{C})) \to \mathrm{coAlg}_R(\mathrm{Ind}(\mathcal{C}))$ is strongly cocontinuous. Denote the right adjoint of U by U^R, which we need to show is cocontinuous. By [23, 4.2.3.3.], the functor $U^{\mathrm{op}} : \mathrm{LMod}_Q(\mathrm{Ind}(\mathcal{C})^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{C})^{\mathrm{op}}$ reflects limits, therefore U reflects colimits. Therefore, given a colimit diagram $F : I \to \mathrm{Ind}(\mathcal{C})$, with colimit $\mathrm{colim}_I F(i)$, we have that $U^R \circ F$ admits $U^R(\mathrm{colim}_I F(i))$ as a colimit if $UU^R \circ F$ admits $UU^R(\mathrm{colim}_I(F(i)))$ as a colimit, which it does since by design $UU^R \simeq Q$ which is cocontinuous by assumption.

It remains to show that $\mathrm{coAlg}(\alpha)$ is strongly cocontinuous. To see this ... □

Remark 4.3.8. Although we were unable to fully prove the above statement, namely strong cocontinuity of $\mathrm{coAlg}(\alpha)$ eluded us, we mention here in passing two ideas which might yield under sufficiently thorough investigation.
The first is that we can try to restrict the adjunction $\mathrm{coEnd} \vdash \mathrm{coAlg}$ even further, in such a way that the proof of theorem 4.3.5 still goes through. In particular, the maximal such reduction would be by restricting it to an adjunction $\mathrm{Pr}^{str,L}_{st,/\mathrm{Ind}(\mathcal{C})} \leftrightarrows \mathrm{coMonad}^L_\star(\mathrm{Ind}(\mathcal{C}))$, where the latter category is the category of cocontinuous comonads, and maps $\alpha : Q \to R$ such that $\mathrm{coAlg}(\alpha)$ is strongly cocontinuous. In order for this to not be too weak for our purposes, one would need to show that given a map of bimodules $a : M \to N$, the associated map of cofree comonads $\mathrm{coFree}(a) : \mathrm{coFrre}(M) \to \mathrm{coFree}(N)$ happens to be in $\mathrm{coMonad}^L_\star(\mathrm{Ind}(\mathcal{C}))$.
The second idea we wish to include here, would involve a lot of work, and is in fact quite different. That is to try and directly show theorem 4.3.5, without decomposing the left adjoint, by trying to imitate the proof of the 1-categorical statement found on the nlab at [27]. The big problem with this strategy given the currently developed theory is that density comonads are, to the best of the author's knowledge, not yet sufficiently developed ∞-categorically. Indeed, the proof of the nlab requires density comonads for maps from the trivial one object category. But, it seems as though the only mention of density comonads (or rather their duals) in the ∞-categorical context is in [11], which only treats the codensity monad of fully faithful inclusions.

5 Derivatives of K-theory

In this section, we collect the various results we accumulated thus far, in order to expose what we can infer about the derivatives of $K : \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{Sp}$. We will also say a quick word about potential further steps.
Before delving into the results, there is a minor obstruction we must still deal with, which is that $K : \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{Sp}$ is not reduced. Specifically, we need a reduced approximation of K-theory, which when precomposed by $\Omega^\infty \simeq \mathrm{Lace}(\mathcal{C}, -) : \mathrm{BiMod}(\mathcal{C}) \to \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ yields K^{cyc}, whose derivatives we have a handle on by theorems 3.4.1 and 3.4.44. In remark 3.4.20, we pointed out that K^{cyc} is the cofiber of K-theory applied to the natural section of $\mathrm{Lace}(\mathcal{C}, M) \to \mathcal{C}$ described in remark 3.3.13. By general abstract nonsense, the cofiber of a section in a stable category is equivalently the fiber of the original map, so that $K^{\mathrm{cyc}}(\mathcal{C}, M) \simeq \mathrm{fib}(K(\mathrm{Lace}(\mathcal{C}, M)) \to K(\mathcal{C}))$. So K^{cyc} is $\widehat{K} \circ \Omega^\infty$, where $\widehat{K}(\mathcal{D} \to \mathcal{C}) := \mathrm{fib}(K(\mathcal{D}) \to K(\mathcal{C}))$.

With this in hand, we have the following result, which is admittedly a bit terse,but everything will be clarified in the course of the proof.

Theorem 5.0.1. *We have the following equivalence*

$$\partial_* \widehat{K} \simeq \mathrm{coBar}(\phi^{-1}(\mathrm{cr}_*(THH(\mathcal{C}, -^{\otimes *})_{hC_*})), \partial_*(U \circ \mathrm{coFree}_{\mathrm{Bimod}(\mathcal{C})}), \boldsymbol{1}).$$

Which we can rephrase as saying that the derivatives of $\widehat{K}$ are Koszul dual to the derivatives of K^{cyc} with respect to the comodule structure on the cooperad given by the derivatives of the comonad which sends a bimodule to the bimodule underlying the corresponding cofree comonad.

Proof. The category $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ is differentiable by proposition 3.2.10 and the discussion preceding it, and Sp is easily seen to be differentiable. The slight modification brought to K-theory was designed to make the functor we are considering reduced and $\widehat{K}$ is finitary by proposition 3.2.6. Therefore, corollary 2.3.12 (and corollary 2.3.2) hold, and imply that we have an equivalence, which can be interpreted as a Koszul duality statement

$$\begin{aligned} \partial_* \widehat{K} &\simeq \mathrm{coBar}(\partial_* \widehat{K}\Omega^\infty, \partial_*(\Sigma^\infty\Omega^\infty), \mathbf{1}) \\ &\simeq \mathrm{coBar}(\partial_* \widehat{K}\,\mathrm{Lace}(\mathcal{C}, -), \partial_*(L_{\mathcal{C}}\,\mathrm{Lace}(\mathcal{C}, -)), \mathbf{1}). \end{aligned}$$

So all that remains is to identify the various pieces in the above result. By the preceding discussion, $\widetilde{K}\,\mathrm{Lace}(\mathcal{C}, -) \simeq K^{\mathrm{cyc}}(\mathcal{C}, -)$. We know the nth homogeneous part of this functor is $THH(\mathcal{C}, -^{\otimes n})_{hC_n}$ by combining 3.4.1 and 3.4.44. From there, the identification $\partial_* \widehat{K}\Omega^\infty \simeq \phi^{-1}(\mathrm{cr}_n(THH(\mathcal{C}, -^{\otimes *})_{hC_*}$ follows from definition 2.1.26, where ϕ^{-1} is the inverse of the equivalence appearing in theorem 2.1.24. The final piece we need to identify is $\Sigma^\infty\Omega^\infty$, which is done in theorem 4.3.5. □

Remark 5.0.2. We remind the reader that the above result is in fact conditional on some caveats described in section 4.3.

The long term goal of this line of thinking would be to use the above result to determine the derivatives of $\widehat{K}$ explicitly, ideally in terms of computationally tractable invariants, which we expect would give powerful methods to understanding the K-theory of $\mathcal{D}$ from the K-theory of $\mathcal{C}$ in presence of a map $\mathcal{D} \to \mathcal{C}$. The probable next step in this direction would be to understand the derivatives of $U \circ \mathrm{coFree}$ and to understand the coaction of $U \circ \mathrm{coFree}$ on K^{cyc}. If this is achieved, we could try to understand the derivatives of this coaction, which would be the coaction with respect to which $\partial_* \widehat{K}$ is identified as a Koszul dual.

We are not able to go much further than what we have accomplished thus far, but we will just say a word about the coaction we are considering, which might make starting further investigation smoother. The $U \circ \mathrm{coFree}$-comodule structure on K^{cyc} comes from the identification $U \circ \mathrm{coFree} \simeq \mathrm{coEnd}(\mathrm{Lace}(\mathcal{C}, -))$ and $K^{\mathrm{cyc}} \simeq \widehat{K}\,\mathrm{Lace}(\mathcal{C}, -)$, more precisely it is obtained from the obvious action of $\mathrm{coEnd}(\mathrm{Lace}(\mathcal{C}, -))$ on $\mathrm{Lace}(\mathcal{C}, -)$ upon whiskering with $\widehat{K}$. This in turn is obtained from the counit $\mathrm{Id} \to L_{\mathcal{C}}\,\mathrm{Lace}(\mathcal{C}, -)$, and then whiskering by $\mathrm{Lace}(\mathcal{C}, -)$ on the left. This is $\mathrm{Lace}(\mathcal{C}, -)$ of a map $\mathrm{Id} \to U \circ \mathrm{coFree}(-)$, which we expect to be the unit of the adjunction $U \dashv \mathrm{coFree}$. Employing the identification $\mathrm{Lace}(\mathcal{C}, -) \simeq \mathrm{coAlg}(\mathrm{coFree}(-))^{\omega}$ which we obtained in the previous section, we get that the coaction we are interested in, is the map

$$\begin{tikzcd}
\mathrm{Lace}(\mathcal{C}, M) \simeq \mathrm{coAlg}(\mathrm{coFree}(M))^{\omega} \arrow[rr] \arrow[dr, "\gamma_M"'] & & \mathrm{coAlg}(\mathrm{coFree}(U(\mathrm{coFree}(M))))^{\omega} \simeq \mathrm{Lace}(\mathcal{C}, U(\mathrm{coFree}(M))) \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

which exhibits γ_M as a $\mathrm{coFree}(U(\mathrm{coFree}(M)))$-comodule classified by the unit of $U \dashv \mathrm{coFree}$ at $\mathrm{coEnd}(\mathrm{Ind}(\gamma_M))$. The desired understanding of this coaction, is to understand it well enough to say something about $\widehat{K}$ of this map.

A Comonads as coendomorphism objects

A.1 Coendomorphism objects

In this appendix, we review the notion of coendomorphism object which we learnt from section 4.1 of [3]. Everything we say can be dualized to endomorphism objects, and there are analogous definitions where left tensored categories are replaced with right tensored ones.

We start with the definition of coendomorphism object.

Definition A.1.1. (Dual of [3, 4.1.1.]) Let $\mathcal{M}$ be an ∞-category left tensored over the ∞-category $\mathcal{C}$ and $x \in \mathcal{M}$. A coendomorphism object for x is an object $c \in \mathcal{C}$ with a map $b : x \to c \otimes x$, such that for any $d \in \mathcal{C}$ the following composition is an equivalence

$$\mathrm{Map}_{\mathcal{C}}(c, d) \xrightarrow{-\otimes x} \mathrm{Map}_{\mathcal{M}}(c \otimes x, d \otimes x) \xrightarrow{b^*} \mathrm{Map}_{\mathcal{D}}(x, d \otimes x).$$

When x admits a coendomorphism object, we write it as $\mathrm{coEnd}(x)$.

Remark A.1.2. In passing we point out that if a coendomorphism object exists, it is unique up to contractible choice. And further that the equivalence characterizing coendomorphism objects is readily seen to be natural in d.

It is immediate from the definition that coendomorphism object classify something which is almost a coaction, and one can wonder if d happens to be a coAlgebra, if there is a sense in which $\mathrm{coEnd}(x)$ classifies coactions of d on x. This is resolved by the following result.

Proposition A.1.3. *(Dual of [3, 4.1.3.]) Let $\mathcal{M}$ be an ∞-category left tensored over the ∞-category $\mathcal{C}$. Suppose $x \in \mathcal{M}$ admits a coendomorphism object $\mathrm{coEnd}(x)$. Then $\mathrm{coEnd}(x)$ admits a coAlgebra structure so that the morphism $b : x \to \mathrm{coEnd}(x) \otimes x$ refines to a left comodule structure. Furthermore for any coAlgebra $Q \in \mathrm{coAlg}(\mathcal{C})$ the canonical map*

$$\mathrm{Map}_{\mathrm{coAlg}(\mathcal{C})}(\mathrm{coEnd}(x), Q) \to \mathrm{LcoMod}_Q(\mathcal{M}) \times_{\mathcal{M}} \{x\}$$

is an equivalence.

Remark A.1.4. We might at times abbreviate the codomain of the equivalence which appears in the above proposition to $\mathrm{LcoMod}_Q(x)$.

Remark A.1.5. The canonical map mentioned in the above proposition is the one which fits in the following diagram, where the bottom map is the composite appearing in the definition of coendomorphism object and with vertical legs the obvious forgetful functors

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Coalg}(\mathcal{C})}(\mathrm{coEnd}(x), Q) \arrow[r] \arrow[d] & \mathrm{LcoMod}_Q(x) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(\mathrm{coEnd}(x), Q) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(x, Q \otimes x). \end{tikzcd}$$

The case of interest to us is described in the following example.

Example A.1.6. Let $f : \mathcal{D} \to \mathcal{C}$ be a left adjoint functor, seen as an object of the ∞-category $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ which is left tensored over $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ via postcomposition. Then a coendomorphism object for f exists, and is given by ff^R with the map $f \to ff^R f$ being the unit whiskered on the right by f. That this map induces the desired equivalence follows from the statement that f is left adjoint to f^R.

Now in proper categorical spirit, having described a construction we ought to understand how it behaves with respect to morphisms, i.e. if it is in any sense functorial. We first do so following [3] with a functoriality which is "very local", and we will supplement this with what one can learn from [19].

Proposition A.1.7. *(Specialization of a dualization of [3, 4.1.9.]) Let $\mathcal{C}, \mathcal{D}_0, \mathcal{D}_1$ be some ∞-categories, and consider* $\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})$ *and* $\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})$ *as left tensored over* $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ *in the standard way via post-composition. Consider a left adjoint $h : \mathcal{D}_0 \to \mathcal{D}_1$, a left adjoint $g : \mathcal{D}_1 \to \mathcal{C}$ and for convenience write $f = gh$. Then there is a unique comonad map $\phi(h) : \mathrm{coEnd}(f) \to \mathrm{coEnd}(g)$ such that for any comonad Q on $\mathcal{C}$ the map*

$$\mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})} \{g\} \xrightarrow{h^*} \mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})} \{f\}$$

corresponds under the equivalence

$$\mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(f), Q) \to \mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})} \{f\}$$

to

$$\mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(g), , Q) \xrightarrow{\phi(h)^*} \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(f), Q).$$

Proof. Because precomposition commutes with postcomposition, we see that $h^* : \mathrm{Fun}(\mathcal{D}_1, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}_0, \mathcal{C})$ is $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ linear. Therefore, the gg^R comodule structure on g is sent by h^* to a gg^R comodule structure on f. This comodule structure is classified by a coAlgebra morphism $\phi(h) : ff^R \to gg^R$.
Now consider a comonad Q on $\mathcal{C}$ and consider the following diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(g), Q) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(f), Q) \arrow[d, "\simeq"] \\
\mathrm{LcoMod}_Q(g) \arrow[r, "h^*"] & \mathrm{LcoMod}_Q(f)
\end{tikzcd}$$

where the top map is the obvious composition involving the inverse of the right vertical map. Because all the maps used to define the top horizontal map are natural in Q the same holds for the top map. But as this is a map of (co)representables, it follows that this is classified by a map $\mathrm{coEnd}(f) \to \mathrm{coEnd}(g)$. To see which map this is, it suffices to take Q to be $\mathrm{coEnd}(g)$, and to see what happens to $\mathrm{Id}_{\mathrm{coEnd}(g)}$. $\square$

Remark A.1.8. Notice that the same proof provides a unique map $\psi(h) : \mathrm{coEnd}(f) \to \mathrm{coEnd}(g)$ such that, for any $M \in \mathrm{Fun}(\mathcal{C}, \mathcal{C})$, whiskering by h seen as a map $\mathrm{Map}(g, M \circ g) \to \mathrm{Map}(f, M \circ f)$ corresponds under the equivalence in the definition of coendomorphism objects to $\psi(h)$. Now using remark A.1.5 we see that $\psi(h) \simeq U\phi(h)$, where U forgets that $\phi(h)$ is a comonad map.

A.2 The "associated comonad" as a functor

The reason we are not satisfied by the above is twofold. On the one hand, it is not completely evident how to use the relative uniqueness of h to make the "associated comonad" construction fully functorial and on the other hand, when we use this construction we will want to have an adjoint for it. This is solved by the following (somewhat abstract) result.

Proposition A.2.1. *([19, 6.30.]) Let $\boldsymbol{C}$ be an $(\infty, 2)$-category that admits Eilenberg-Moore objects. There is an adjunction*

$$\mathrm{End} : \mathrm{Fun}([1], \boldsymbol{C})^R \rightleftarrows \mathrm{Mon}(\boldsymbol{C})^{\mathrm{op}} : \mathrm{Alg},$$

where End *sends a right adjoint to its associated monad and* Alg *sends a monad to its Eilenberg-Moore object and restricts to an equivalence* $\mathrm{Mon}(\boldsymbol{C})^{\mathrm{op}} \simeq \mathrm{Fun}([1], \boldsymbol{C})^{\mathrm{mon}}$.
Furthermore the unit $Z \to \mathrm{coAlg}(\mathrm{coEnd}(g))$ for $g : Z \to X$ classifies the canonical coaction of $\mathrm{coEnd}(g)$ *on g.*

All examples of interest to us are when $\mathbf{C}$ is taken to be a subcategory of the $(\infty, 2)$-category of $(\infty, 1)$-categories, with all 1 and 2-cells reversed, denoted by $\mathbf{C}^{\mathrm{co,op}}$. In the maximal case, we get the following specification of the above result.

Proposition A.2.2. *There is an adjunction*

$$\mathrm{coEnd} : \mathrm{Fun}([1], \mathrm{Cat})^L \rightleftarrows \mathrm{coMonad}(\mathrm{Cat}) : \mathrm{coAlg}$$

where coEnd *sends a left adjoint to its associated comonad and* oAlg *sends a comonad to its category of coAlgebras.*

Fixing some ∞-category $\mathcal{C}$, this adjunction can be suitably restricted to the following adjunction which is the case we use.

Theorem A.2.3. *There is an adjunction*

$$\mathrm{coEnd} : (\mathrm{Cat}_{/\mathcal{C}})^L \rightleftarrows \mathrm{coMonad}(\mathcal{C}) : \mathrm{coAlg}\,.$$

Remark A.2.4. We will use the same notation for the functors coEnd and coAlg, even if we consider a subcategory of the whole category of ∞-categories.

Now the problem is, that given a commutative triangle over an ∞-category $\mathcal{C}$

$$\begin{tikzcd} \mathcal{D}_0 \arrow[rr, "h"] \arrow[dr, "f"'] & & \mathcal{D}_1 \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

there are two map of comonads $ff^R \to gg^R$, on the one hand $\phi(h)$ the one obtained using [3] and on the other hand $\mathrm{coEnd}_{\mathrm{Map}}(h)$ the one obtained by the functor coEnd. The subscript Map serves to lift the potential ambiguity concerning whether we see h as an object or a morphism in the arrow category of ∞-categories. Our next purpose is to show that these are in fact the same.

Proposition A.2.5. *Given a commutative triangle of left adjoints $f \simeq gh$ over $\mathcal{C}$ as above, the maps $\phi(h)$ and $\mathrm{coEnd}_{\mathrm{Map}}(h)$ are in fact the same. In particular the association sending $f : \mathcal{D}_0 \to \mathcal{C}$ to ff^R and sending commutative triangle $f \simeq gh$ to $\phi(h)$ can be made into a functor.*

Proof. Because $\phi(h)$ is equipped with a universal property, this amounts to showing that $\mathrm{coEnd}_{\mathrm{Map}}(h)$ has the same property. For this we use that the latter map fits in the following commutative diagram by the statement that coEnd is left adjoint to coAlg

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(g), Q) \arrow[r] \arrow[d, "\mathrm{coEnd}_{\mathrm{Map}}(h)^*"'] & \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{C}}}(g, \mathrm{coAlg}(Q)) \arrow[d, "h^*"] \\ \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(f), Q) \arrow[r] & \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{C}}}(f, \mathrm{coAlg}(Q)). \end{tikzcd}$$

Now by (dual of) the unenriched version of [19, 3.39], we can expand this diagram to

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(g), Q) \arrow[r] \arrow[d, "\mathrm{coEnd}_{\mathrm{Map}}(h)^*"'] & \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{C}}}(g, \mathrm{coAlg}(Q)) \arrow[r] \arrow[d, "h^*"] & \mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})} \{g\} \arrow[d, "h^*"] \\ \mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(f), Q) \arrow[r] & \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{C}}}(f, \mathrm{coAlg}(Q)) \arrow[r] & \mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_0, \mathcal{C})} \{f\} \end{tikzcd}$$

where the identification of the right most leg follows from the fact that right horizontal equivalences provided by [19, 3.39] are over $\mathrm{Fun}(\mathcal{D}_i, \mathcal{C})$.
We notice that the horizontal composite must be the "canonical" one from proposition A.1.3 as both maps are natural, so it suffices to observe what happens when we take $Q = \mathrm{coEnd}(g)$ (resp. $Q = \mathrm{coEnd}(f)$). This can be understood as we know that the unit of $\mathrm{coEnd} \dashv \mathrm{coAlg}$ classifies the canonical $\mathrm{coEnd}(g)$-coaction. $\square$

Remark A.2.6. Technically, it was outside of the scope of this project to understand [19, 3.39.] with sufficient clarity for us to be wholly confident in the above proof. In theory, the equivalence we use $\mathrm{Map}_{\mathrm{Cat}_{/\mathcal{C}}}(g, \mathrm{coAlg}(Q)) \to \mathrm{LcoMod}_Q(\mathrm{Fun}(\mathcal{D}_1, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}_1,\mathcal{C})} \{g\}$ could be exotic in some way, and thus the horizontal composite in the above proof could be something other than the expected canonical map.
We have opted to leave the proof nonetheless, on the one hand because it is almost certain that this problem does not actualize and on the other because we do not actually use the above theorem in any way in the main body of the text.

A.3 Making $\mathrm{Nat}(\mathrm{coEnd}(g), M) \to \mathrm{Nat}(g, M \circ g)$ natural

This section is deeply indebted to Prof. Heine, who kindly gave essentially all of the ideas of the proof we present here. Furthermore, for the sake of full intellectual transparency, some details are not present because the author was unable to figure them out, and because they were obvious to the author. However, since the result and the bulk of the argument comes from Prof. Heine, we have decided to trust his word.

What we have achieved so far almost makes it look like we have a natural map

$$\mathrm{Map}_{\mathrm{coMonad}(\mathcal{C})}(\mathrm{coEnd}(g), Q) \to \mathrm{LcoMod}_Q(g),$$

which upon forgetting the comonad structure would give a natural map

$$\mathrm{Nat}(\mathrm{coEnd}(g), Q) \to \mathrm{Nat}(g, Q \circ g).$$

This is almost achieved by consulting the proof of proposition A.1.7 along with proposition A.2.5. However, at best, the idea as just described can only provide a "naturality square" for

$$\mathrm{Nat}(\mathrm{coEnd}(g), M) \to \mathrm{Nat}(g, M \circ M),$$

rather than the stronger statement involving all higher coherences that this arrow is in fact natural. But we in fact need full naturality to prove theorem 4.3.4. Since this comes at no extra cost, we work at the generality of a 2-category $\mathbf{C}$ and object X if $\mathbf{C}$. We introduce the following convenient notation

Notation A.3.1. Let $\mathbf{C}$ be an $(\infty, 2)$-category, and X and Y be objects in $\mathbf{C}$. We denote by $\mathbf{C}(Y, X)$ the $(\infty, 1)$-category of morphisms between Y and X. Now given f and g morphisms from Y to X, we denote by $\mathrm{Nat}_{\mathbf{C}(Y,X)}(f, g)$ the space of maps between f and g. We will at times suppress the subscript from this notation, writing it simply as $\mathrm{Nat}(f, g)$.

It turns out that the biggest difficulty is in defining the functor $(\mathbf{C}^L_X)^{\mathrm{op}} \times \mathbf{C}(X, X) \to \mathcal{S}$ which maps a pair $(g : Y \to X, M : X \to X)$ to $\mathrm{Nat}_{\mathbf{C}(X,X)}(g, M \circ g)$, and so this is our first result.

Proposition A.3.2. *Let $\mathbf{C}$ be an $(\infty, 2)$-category, and let X be an object in $\mathbf{C}$. Then there is a functor $\rho : (\mathbf{C}^L_X)^{\mathrm{op}} \times \mathbf{C}(X, X) \to \mathcal{S}$ which maps $(g : Y \to X, M : X \to X)$ to $\mathrm{Nat}_{\mathbf{C}(Y,X)}(g, M \circ g)$.*

The proof will proceed by applying a more general construction, which in turn requires the following lemma.

Lemma A.3.3. *Given a coCartesian fibration $p : T \to S$ and an ∞-category K, there is a coCartesian fibration $T^K \to S$, defined as the pullback of the induced map $p_* : \mathrm{Fun}(K, T) \to \mathrm{Fun}(K, S)$ along the diagonal map $\Delta : S \to \mathrm{Fun}(K, S)$.*

Proof. This follows by directly combining the duals of [22, 3.1.2.1.] (applying $\mathrm{Fun}(K, -)$ to a coCartesian fibration yields a coCartesian fibration) and [22, 2.4.2.3.] (coCartesian fibrations are stable under pullback).. □

With this in hand we state and prove our key technical result.

Proposition A.3.4. *Let $p : T \to S$ be a coCartesian fibration. Let $\alpha : S \to T$ be a coCartesian section of p (i.e. sending every edge to a p-coCartesian one) and let β be any section of p. There is a coCartesian fibration $\gamma : S \times_T T^{\Delta^1} \times_T S \to S$, which classifies the functor sending s to* $\mathrm{Map}_{T_s}(\alpha(s), \beta(s))$.

Proof. The construction starts by considering the pullback

$$\begin{tikzcd} S \times_T T^{\Delta^1} \arrow[r] \arrow[d] & T^{\Delta^1} \arrow[d, "\mathrm{dom}"] \\ S \arrow[r, "\alpha"'] & T \end{tikzcd}$$

and then further consider the pullback

$$\begin{tikzcd} S \times_T T^{\Delta^1} \times_T S \arrow[r] \arrow[dd] & S \times_T T^{\Delta^1} \arrow[d] \\ & T^{\Delta^1} \arrow[d, "\mathrm{cod}"] \\ S \arrow[r, "\beta"'] & T. \end{tikzcd}$$

We are interested in the map $\gamma : S \times_T T^{\Delta^1} \times_T S \to S$ which appears in the second diagram. Because β need not preserve coCartesian fibrations, γ cannot be considered a pullback in the category of coCartesian fibrations over the base S. So we need to show that it is a coCartesian fibration directly, the result will then follow from direct inspection using that applying $\mathrm{dom}_T^{\Delta^1} \to T$ to a an edge which is cod-coCartesian yields a p-coCartesian edge. ☐

Proof. (of proposition A.3.2) Since we are trying to construct a functor to spaces, we can instead construct a left or right fibration classifying it. We will construct a left fibration $Z \to (\mathbf{C}_X^L)^{\mathrm{op}} \times \mathbf{C}(X, X)$.
Consider the Cartesian fibration $\mathbf{C}_{//X} \to \mathbf{C}$ which classifies the functor

$$\mathbf{C}(-, X)^{\mathrm{op}} : \mathbf{C}^{\mathrm{op}} \xrightarrow{\mathbf{C}(-,X)} \mathrm{Cat}_\infty \xrightarrow{\mathrm{op}} \mathrm{Cat}_\infty$$

And consider also $\mathbf{C}_{/X} \to \mathbf{C}$, the right (in particular Cartesian) fibration classifying

$$\mathbf{C}(-, X)^{\simeq} : \mathbf{C}^{\mathrm{op}} \to \mathcal{S}.$$

The inclusion of the maximal groupoid core naturally gives rise to a natural map

$$\mathbf{C}(-, X)^{\simeq} \to (\mathbf{C}(-, X)^{\simeq})^{\mathrm{op}} \to \mathbf{C}(-, X)^{\mathrm{op}},$$

which unstraightens to a map $\mathbf{C}_{/X} \to \mathbf{C}_{//X}$ of Cartesian fibrations over $\mathbf{C}$.
Suitably restricting the composition functor, we get, for any Y in $\mathbf{C}$, a natural transformation

$$\mathbf{C}(-, X) \times \mathbf{C}(X, Y) \to \mathbf{C}(-, Y)$$

of functors $\mathbf{C} \to \mathrm{Cat}$. Taking $(-)^{\mathrm{op}}$ of this natural transformation gives a natural map

$$\mathbf{C}(-, X)^{\mathrm{op}} \times \mathbf{C}(X, Y)^{\mathrm{op}} \to \mathbf{C}(-, Y)^{\mathrm{op}}.$$

This unstraightens to a map $\kappa : \mathbf{C}_{//X} \times \mathbf{C}(X, Y)^{\mathrm{op}} \to \mathbf{C}_{//Y}$ of Cartesian fibrations over $\mathbf{C}$.
We now can apply the above proposition. First we fix our objects: taking $S = (\mathbf{C}_{/X})^{\mathrm{op}} \times \mathbf{C}(X, X)$ and $T = (\mathbf{C}_{/X})^{\mathrm{op}} \times_{\mathbf{C}^{\mathrm{op}}} (\mathbf{C}_{//X})^{\mathrm{op}} \times \mathbf{C}(X, X)$. There is a clear map $p : T \to S$, and we can now define α and β. To construct α, we need a map $S \to \mathbf{C}(X, X)$ and a map $S \to (\mathbf{C}_{/X})^{\mathrm{op}} \times_{\mathbf{C}^{\mathrm{op}}} (\mathbf{C}_{//X})^{\mathrm{op}}$. For the former, we can take the natural projection, and for the latter we take the map given by universal property of $(\mathbf{C}_{/X})^{\mathrm{op}} \times_{\mathbf{C}} (\mathbf{C}_{//X})^{\mathrm{op}}$ by the identity maps and the map $\mathbf{C}_{/X} \to \mathbf{C}_{//X}$ we described in the

first paragraph. The universal construction of α ensures that every edge of S is sent to a p-coCartesian one of T. For β, first consider the composition

$$(\mathbf{C}_{/X})^{\mathrm{op}} \times \mathbf{C}(X,X) \xrightarrow{\iota \times \mathbf{C}(X,X)} (\mathbf{C}_{//X})^{\mathrm{op}} \times \mathbf{C}(X,X) \xrightarrow{\kappa^{\mathrm{op}}} (\mathbf{C}_{//X})^{\mathrm{op}}.$$

It follows by direct inspection that this map is suitably over $\mathbf{C}^{\mathrm{op}}$, and that it therefore gives a map to T, which is the map we take to be β. For both of these maps, it is sufficiently clear that they are sections to p.
We now have a coCartesian fibration $\gamma : S \times_T T^{\Delta^1} \times_T S$, whose fibers we wish to understand. By the previous proposition, to do this suitably it suffices to understand the fibers of p. To this end fix $(g : Y \to X, M : X \to X)$ an object of S, by direct inspection, the fiber of p at the above is the fiber of $\mathbf{C}^{\mathrm{op}}_{//X} \to \mathbf{C}^{\mathrm{op}}$ at Y. Since $\mathbf{C}_{//X} \to \mathbf{C}$ classifies the functor $\mathbf{C}^{\mathrm{op}} \to \mathrm{Cat}$ sending Y to $\mathbf{C}(Y,X)^{\mathrm{op}}$, we get that the fibers of p are $(\mathbf{C}(Y,X)^{\mathrm{op}})^{\mathrm{op}} \simeq \mathbf{C}(Y,X)$.
It follows that γ classifies the functor sending $(g : Y \to X, M : X \to X)$ to

$$\mathbf{C}(Y,X)(\alpha(g : Y \to X, M : X \to X), \beta(g : Y \to X, M : X \to X)).$$

We engineered α and β so that in the fiber of $(g : Y \to X, M : X \to X)$ (which is where we are considering α and β) we get $\alpha(g : Y \to X, M : X \to X) = g$ and $\beta(g : Y \to X, M : X \to X) = M \circ g$ and thus γ classifies the desired functor. $\square$

With this hand, we can prove the following result which is the one we want.

Lemma A.3.5. *There is a natural equivalence* $\alpha : \mathrm{Nat}_{\boldsymbol{C}(X,X)}(\mathrm{coEnd}(g), M) \to \mathrm{Nat}_{\boldsymbol{C}(Y,X)}(g, M \circ g)$ *of functors* $(\boldsymbol{C}^L_X)^{\mathrm{op}} \times \boldsymbol{C}(X,X) \to \mathcal{S}$.

Proof. For any fixed $g \in (\mathbf{C}^L_X)^{\mathrm{op}}$, the restriction $\rho(g,-)$ of the functor constructed previously is corepresentable, which is to say in the essential image of the coYoneda embedding $\mathbf{C}(X,X) \to \mathrm{Fun}(\mathbf{C}(X,X), \mathcal{S})$. This follows from the fact that if g^R is the right adjoint of g, then $- \circ g^R$ is left adjoint to $- \circ g$, and so we have a natural equivalence

$$\mathrm{Nat}(g \circ g^R, M) \simeq \mathrm{Nat}(g, M \circ g)$$

showing the desired corepresentability. By fully faithfulness of the Yoneda embedding, this implies that the currying $(\mathbf{C}^L_X)^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{C}(X,X), \mathcal{S})$ of ρ factors through $\mathbf{C}(X,X)$.
Since this functor sends g to gg^R, we are justified in calling it coEnd, we get an equivalence between $g \mapsto \rho(g,-)$ and $g \mapsto \mathrm{Nat}(\mathrm{coEnd}(g), -)$, which upon currying gives us the desired equivalence. $\square$

Remark A.3.6. A priori, there is an ambiguity concerning how the coEnd functor in the above proof compares with the one from the previous subsection which adjoint to coAlg.

References

[1] Clark Barwick, Saul Glasman, Akhil Mathew, and Thomas Nikolaus. K-theory and polynomial functors. Preprint, arXiv:2102.00936 [math.KT] (2021), 2021.

[2] Kristine Bauer, Matthew Burke, and Michael Ching. Tangent infinity-categories and Goodwillie calculus. Preprint, arXiv:2101.07819 [math.CT] (2021), 2021.

[3] Max Blans and Thomas Blom. On the chain rule in Goodwillie calculus. Preprint, arXiv:2410.20504 [math.AT] (2025), 2025.

[4] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic K-theory. *Geom. Topol.*, 17(2):733–838, 2013.

[5] Lukas Brantner, Ricardo Campos, and Joost Nuiten. PD Operads and Explicit Partition Lie Algebras. Preprint, arXiv:2104.03870 [math.AG] (2021), 2021.

[6] Lukas Brantner and Akhil Mathew. Deformation theory and partition Lie algebras. *Acta Math.*, 235(1):1–148, 2025.

[7] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity. Preprint, arXiv:2009.07224 [math.KT] (2020), 2020.

[8] Baptiste Calmès, Emanuele Dotto, Yonatan Harpaz, Fabian Hebestreit, Markus Land, Kristian Moi, Denis Nardin, Thomas Nikolaus, and Wolfgang Steimle. Hermitian K-theory for stable ∞-categories. I: Foundations. *Sel. Math., New Ser.*, 29(1):269, 2023. Id/No 10.

[9] Bjørn Ian Dundas, Thomas G. Goodwillie, and Randy McCarthy. *The Local Structure of Algebraic K-Theory.* Algebra and Applications. Springer, Germany, 2013.

[10] Alexander I. Efimov. K-theory and localizing invariants of large categories. Preprint, arXiv:2405.12169 [math.KT] (2024), 2024.

[11] Emmanuel Dror Farjoun and Sergei Olegovich Ivanov. Bousfield-kan completion as a codensity ∞-monad. Preprint, arXiv:2507.08414 [math.AT] (2025), 2025.

[12] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in ∞-categories. *Doc. Math.*, 22:1225–1266, 2017.

[13] Thomas G. Goodwillie. Calculus. I. The first derivative of pseudoisotopy theory. *K-Theory*, 4(1):1–27, 1990.

[14] Thomas G. Goodwillie. Calculus. II. Analytic functors. *K-Theory*, 5(4):295–332, 1991/92.

[15] Thomas G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711, 2003.

[16] Yonatan Harpaz, Thomas Nikolaus, and Victor Saunier. Trace methods for stable categories I: The linear approximation of algebraic K-theory. Preprint, arXiv:2411.04743 [math.AT] (2024), 2024.

[17] Rune Haugseng. On lax transformations, adjunctions, and monads in $(\infty, 2)$-categories. *High. Struct.*, 5(1):244–281, 2021.

[18] Fabian Hebestreit, Andrea Lachmann, and Wolfgang Steimle. The localisation theorem for the K-theory of stable ∞-categories. *Proc. R. Soc. Edinb., Sect. A, Math.*, 154(6):1749–1785, 2024.

[19] Hadrian Heine. A monadicity theorem for higher algebraic structures. Preprint, arXiv:1712.00555 [math.CT] (2017), 2017.

[20] Claudius Heyer and Lucas Mann. 6-functor Formalisms and Smooth Representations. Preprint, arXiv:2410.13038 [math.CT] (2024), 2024.

[21] Marc Hoyois, Sarah Scherotzke, and Nicolò Sibilla. Higher traces, noncommutative motives, and the categorified Chern character. *Adv. Math.*, 309:97–154, 2017.

[22] Jacob Lurie. *Higher topos theory*, volume 170 of *Ann. Math. Stud.* Princeton, NJ: Princeton University Press, 2009.

[23] Jacob Lurie. Higher algebra. https://www.math.ias.edu/ lurie/, 2017.

[24] Jacob Lurie. Elliptic Cohomology I: Spectral Abelian Varieties. https://www.math.ias.edu/ lurie/, 2018.

[25] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[26] nLab authors. Euler characteristic. `https://ncatlab.org/nlab/show/Euler+characteristic`, December 2025. Revision 40.

[27] nLab authors. free monad. `https://ncatlab.org/nlab/show/free+monad`, January 2026. Revision 20.

[28] Joost Nuiten. On straightening for Segal spaces. *Compos. Math.*, 160(3):586–656, 2024.

[29] Maxime Ramzi. The additivity of traces in stable ∞-categories. Preprint, arXiv:2109.01512 [math.KT] (2021), 2021.

[30] Maxime Ramzi. Separability in homotopy theory and topological hochschild homology. PhD thesis of the author, available at: https://sites.google.com/view/maxime-ramzi-en, 2024.

[31] Sam Raskin. On the Dundas-Goodwillie-McCarthy theorem. Preprint, arXiv:1807.06709 [math.AT] (2018), 2018.

[32] Emily Riehl and Dominic Verity. Homotopy coherent adjunctions and the formal theory of monads. *Adv. Math.*, 286:802–888, 2016.

[33] Victor Saunier. Trace methods for stable ∞-categories. PhD thesis of the author, available at: https://victor-saunier.fr/, 2025.

[34] Wolfgang Steimle. On the universal property of Waldhausen's K-theory. Preprint, arXiv:1703.01865 [math.KT] (2017), 2017.

[35] Dév Vorburger. Author's home page. `https://www.xn--dv-bja.dev/`, May 2025.

[36] Tashi Walde. Homotopy coherent theorems of Dold-Kan type. *Adv. Math.*, 398:53, 2022. Id/No 108175.

[37] Charles A. Weibel. *An introduction to homological algebra. 1st pbk-ed*, volume 38 of *Camb. Stud. Adv. Math.* Cambridge: Cambridge Univ. Press, 1st pbk-ed. edition, 1995.