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# Goodwillie calculus and derivatives of the identity

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# Contents

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I leave this as a riddle for anybody daring enough to try, as we now embark on a different series of riddles regarding Goodwillie calculus.

# Introduction

The goal of this project is to make sense of the sentence "the derivatives of the identity functor for spaces is the theory of spectral lie algebras". This statement makes sense via the language of Goodwillie calculus, a topic in topology initiated by Goodwillie in his series of groundbreaking papers [11], [12] and [13]. We will spend sections §1 and §2 building up the necessary material to understand the definition of the derivative of the identity and then in section §3 we will cover a somewhat heterogeneous collection of topics which serve to make [6] approachable enough to understand the sentence which motivated this project.

We assume that the reader is comfortable with basic topology at the level of Hatcher's book on the subject [14]. The other prerequisites for this project are all adjacent to algebraic topology, such as categorical maturity and the language of simplicial sets. The former is acquired simply through practice, and for the latter we recommend the book by Goerss and Jardine [10] on the subject. We also assume that the reader is comfortable with accepting the result from [21] and [20], although the formal prerequisites from these books is minimal. In particular all of the categorical notions we use are $\infty$-categorical, and the reader should be willing to at least accept these on faith. It is however not necessary to understand the details of how we obtain the result in the $\infty$-categorical setting.
In particular, whenever we call something a category, we mean an $\infty$-category, and more specifically, we work with the model of quasi-categories.

Relative to ordinary calculus, our approach to Goodwillie calculus does like transfer maps and goes the wrong way, starting with the definition of the Taylor tower and only then going on to the define the derivative. In section §1 we construct the Taylor tower of a functor. In subsection §1.1 we define an $n$-excisive functor, which will play the role analogous to that of polynomials of degree $n$ in ordinary calculus. Then in subsection §1.2 we prove the following theorem.

**Theorem 0.0.1.** *(Theorem 6.1.1.10. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. Then the natural inclusion functors* $\mathrm{Exc}^n(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ *admits a left adjoint which we denote by $P_n$. Further these functors are left exact (i.e. preserve finite limits).*

Which corresponds in ordinary calculus to the existence of degree $n$ Taylor approximations. We take a first step in defining the derivative from the Taylor tower in subsection §1.3, which is a basic study of the difference between a degree $n$ and $(n-1)$-approximation of a functor, thus giving an object corresponding to $\frac{f^{(n)(0)}}{n!}x^n$ in ordinary calculus. The main result in our study is the following theorem.

**Theorem 0.0.2.** *(Theorem 6.1.2.4. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ a differentiable category. Then, for all n, we have a diagram in* $\mathrm{Fun}(\mathrm{Fun}_*(\mathcal{C},\mathcal{D}),\mathrm{Fun}_*(\mathcal{C},\mathcal{D}))$*:*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[d] \\ K_n \arrow[r] & R_n \end{tikzcd} .$$

*The functors $P_n, P_{n-1}$ are the n-excisive approximations of theorem 1.2.1 implicitly postcomposed with the inclusions* $\mathrm{Exc}^n(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$*, and $R_n$ and $K_n$ satisfy the following properties.*

*(i) For every reduced functor $F : \mathcal{C} \to \mathcal{D}$, $K_n(F)$ carries every object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

*(ii) For every reduced functor $F : \mathcal{C} \to \mathcal{D}$ we have that $R_n(F)$ is n-homogeneous.*

*(iii) The functor $R_n$ is left exact.*

*(iv) If $F : \mathcal{C} \to \mathcal{D}$ is $(n-1)$-excisive, then $R(F)$ carries every object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

The above theorem is interesting in its own right because it implies that $D_nF = \mathrm{Fib}(P_nF \to P_{n-1}F)$ is a loop space under some nice assumption. And any structure we can give the layers of the Taylor tower can be used to study $F$. However, for the goals of this project, the main point of the above theorem is the following corrolary which we also prove in subsection §1.3.

**Corollary 0.0.3.** *(Corollary 6.1.2.9. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category. Then composition by $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces for any integer $n$ an equivalence*

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).$$

Indeed this result will be instrumental in the proof of the main theorem of subsection §2.2.

**Theorem 0.0.4.** *(6.1.4.7. in [21] Let $\mathcal{C}$ be a pointed category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category. Then we have a fully faithfull embedding*

$$\mathrm{cr}_{(n)} : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}).$$

*The essential image of $\mathrm{cr}_{(n)}$ is the full subcategory $\mathrm{SymFun}^n_{lin}(\mathcal{C}, \mathcal{D})$ of those functors $E : \mathcal{C}^{(n)} \to \mathcal{D}$ whose underlying functor $E : \mathcal{C}^n \to \mathcal{D}$ is multilinear.*

However before being able to study this theorem, we start section §2, whose end goal is to define the derivative, by studying the basics of multivariable Goodwillie calculus in subsection §2.1. The main output of this study is the following definition.

**Definition 0.0.5.** (Construction 6.1.3.20. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed category with finite limits, and $F : \mathcal{C} \to \mathcal{D}$. We have a functor $q : \mathcal{C}^n \to \mathcal{C}$ which maps an $n$-tuple to the coproduct of these objects. The functor $\mathrm{cr}_n := \mathrm{Red}(F \circ q) : \mathcal{C}^n \to \mathcal{D}$ is defined, this is what we call the *nth-cross effect*.

This then gives way to the proof of the above theorem in subsection §2.2. After which we provide the final ingredient for the definition of the derivative in subsection §2.4. This final ingredient is the following theorem from subsection §2.3.

**Theorem 0.0.6.** *(Proposition 6.2.3.21. and Corollary 6.2.3.22. in [21]) Let $\{\mathcal{C}_i\}_{i\in I}$ be a finite collection of pointed differentiable category and let $\mathcal{D}$ be a differentiable category. Then the construction $f \mapsto \Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{i\in I} \Sigma^\infty_{\mathcal{C}_i}$ defines an equivalence $\phi : \mathrm{Exc}_\star(\prod_{i\in I} \mathrm{Sp}(\mathcal{C}_i), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_\star(\prod_{i\in I} \mathcal{C}_i, \mathcal{D})$.*

The proof of this theorem goes through a hands-on definition of the first derivative in subsubsection §2.3.1 and a more detailed study of stable categories in §2.3.2.

At this point, we can now understand half of the sentence “the derivatives of the identity functor for spaces is the theory of spectral lie algebras” which guides this project. In section §3, although we won’t fully elucidate the second half of the sentence, we discuss enough material for the interested reader to finish deciphering the guiding light of the project by reading [6].
We start this section by studying (co)ends in subsection §3.1, a categorical tool which is used in to define the bar construction for operads used by Ching to study the derivatives of the identity. We don’t prove any specifically useful result in this section, but instead simply try to develop some intuition for this construction. We then study bar constructions in subsection §3.2, once again with the goal of developing intution more so than trying to prove any specific result. We study two different and rather general bar constructions, before studying the specific case of differential graded algebras which serves as the key inspiration for the bar construction for operads. Then in subsection §3.3 we give a brief introduction to Koszul duality, which is used in Ching’s paper [6] to characterize the derivatives of the identity as Koszul dual to the commutative operad in spectra, and thus in analogy to the classical case, deserving of the name “spectral lie operad”. The study of Koszul duality is heavily dependent on the bar construction we discussed in the previous section.
After these three, in some sens relaxed, subsections we move on to subsection §3.4, which is the final part of this project, where we compute the derivatives of the identity. Our final result is the following, which we prove almost in full.

**Theorem 0.0.7.** *(Corollary 14 in [3]) There is an equivalence of $\Sigma_n$-spectra*

$$\partial_n(\mathrm{Id}_{\mathcal{S}_*}) \simeq \mathrm{Map}(\Sigma^\infty \Delta_n, \mathbb{S}).$$

## Convention

We adopt the philosophy that, whenever possible, a diagram is worth a thousand words, even when it might be quite intimidating. In order for this to be reasonable, we often omit naming all the arrows appearing in a diagram, and let context assist in understanding. There are some naming conventions we use at times to help with readability. The most important are that, if a map is obtained by universal property, and so is unique (up to contractible choice), we often denote it by !. Inclusions are often denoted by $\iota$, potentially with a subscript denoting the domain and projections are often denoted by $\pi$, with a potential subscript used to denote the codomain. Diagonal morphisms, that is morphisms $X \to X^{\times n}$ obtained by universal property using the identity, are denoted by $\Delta$.

The set $[n]$ is the set of positive integers $\{0, 1, ..., n\}$. We will sometimes view this as a finite poset, as a 1-category or as an $\infty$-category. We allow ourselves to do this without further comments at times.

Although this project sticks very close to the notation and language of Lurie's books [20] and [21], we have one major difference. For ease of reference we discuss this point in a remark

*Remark* 0.0.8. We call a functor $F : A \to B$ final if the natural map $\varinjlim_B D \to \varinjlim_A D \circ F$ is an equivalence, where $D : B \to \mathcal{C}$ is a diagram. Dually, if precomposition by $F$ preserves limits, we call $F$ initial. This is not the nomenclature Lurie uses, but instead the one advocated for example by the nLab. But we still use the term cofinal for subcategories/subsimplicial sets such that every object in the large category maps to an object of the subcategory. And similarly we will still call reasoning using this strategy "cofinality arguments".

# 1 Constructing the Taylor tower

We start this project by exploring parts of chapter 6 of [21] in order to learn the basics of Goodwillie calculus. The goal of functor calculus in general is to create tools inspired by classical calculus, in Euclidean space or on manifolds, in the context of functors between categories. Goodwillie calculus is one such example of a functor calculus, originally developped in the three foundational papers of Goodwillie: [11], [12] and [13]. The analogy with ordinary calculus is explored in [2].

In this section specifically, we wish to construct something analogous to the Taylor series of ordinary calculus inside of a functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Unsuprisingly this is not done in absolute generality, but instead with some assumptions on $\mathcal{C}$ and $\mathcal{D}$. we will require that $\mathcal{C}$ admits all finite colimits and a final object and that $\mathcal{D}$ is differentiable, whose definition is

**Definition 1.0.1.** (Definition 6.1.1.6. in [21]) A category $\mathcal{D}$ is called differentiable if:

(i) $\mathcal{D}$ admits all finite limits,

(ii) $\mathcal{D}$ admits sequential colimits, i.e. colimits for $\mathrm{N}(\mathbb{Z})$-shaped diagrams.

(iii) The colimit functor $\varinjlim \mathrm{Fun}(\mathrm{N}(\mathbb{Z}), \mathcal{D}) \to \mathcal{D}$ is left exact, i.e preserves finite limits.

Key examples of differentiable categories are stable categories with countable coproducts, such as the category of spectra (see example 6.1.1.7. in [21]) and compactly generated categories (see example 6.1.1.9. in [21]), including key examples such as the category of spaces or of infinity categories. For more about this important class of $\infty$-categories, we invite the reader to consult section 5.5.7 of [20]

In order to construct something akin to a Taylor series, we need a notion of degree $n$ polynomial functor, and a canonical way to approximate a functor by a degree $n$-functor. We do the former in subsection §1.1 and the latter in subsection §1.2. In this section we also assemble all of these approximations into a tower which will be our “Taylor Series”. Then in subsection §1.3 we establish a result comparing the degree $n$ approximation with the degree $n-1$ approximation.

## 1.1 Excisive Functors

An excisive functor should be imagined as the object analogous to polynomials, and an $n$-excisive will be analogous to degree $n$-polynomials. Before defining these, we need a few definitions.

**Definition 1.1.1.** (Notation 6.1.1.1. in [21]) Given a (finite) set $S$, we have a 1-category $\mathcal{P}(S)$ of subsets corresponding to the poset $(2^S, \subset)$. We can see this as an $\infty$-category by taking the nerve. We denote by $\mathcal{P}_{\leq i}(S)$ the full subcategory whose objects are the subset of cardinality less than or equal to $i$. We define $\mathcal{P}_{>i}(S)$ similarly.

**Definition 1.1.2.** (Definition 6.1.1.2. in [21]) For a category $\mathcal{C}$, an $n$-cube in $\mathcal{C}$ is an $\mathrm{N}(\mathcal{P}(S))$-shaped diagram in $\mathcal{C}$, where $S$ is a finite set of cardinality $n$. We may also call these $S$ cubes. We call the functor category $\mathrm{Fun}(\mathrm{N}(\mathcal{P}(S)), \mathcal{C})$ the category of $n$-cubes in $\mathcal{C}$.

If we decompose $S$ as $T_- \sqcup T \sqcup T_+$, we can obtain a $T$-cube from an $S$-cube $X$ by mapping $T_0 \subset T$ to $X(T_- \sqcup T_0)$. The $T$-cubes obtained this way are called $T$-faces of $X$.

To justify the nomenclature of cubes, it is instructive to observe what happens for [1] and [2]:

$$
\begin{tikzcd}
 & \emptyset \arrow[r] \arrow[dl] & \{0\} \arrow[dl] \\
\{1\} \arrow[r] & \{0,1\} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \emptyset \arrow[rr] \arrow[dl] \arrow[dd] & & \{0\} \arrow[dl] \arrow[dd] \\
\{1\} \arrow[rr] \arrow[dd] & & \{0,1\} \arrow[dd] & \\
 & \{2\} \arrow[rr] \arrow[dl] & & \{0,2\} \arrow[dl] \\
\{1,2\} \arrow[rr] & & \{0,1,2\} &
\end{tikzcd}
.
$$

**Definition 1.1.3.** (Definition 6.1.1.2. in [21]) An $n$-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ is called Cartesian if the obvious map

$$X(\emptyset) \to \varprojlim_{\emptyset \neq S \subset [n]} X(S)$$

is an equivallence.
We call an $n$-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ strongly coCartesian if it is a left Kan extension of its restriction to $\mathrm{N}(\mathcal{P}([n])_{\leq 1})$.

It is obvious how to define the dual notions of coCartesian and strongly cartesian, however as we won't be needing these notions, we won't make them explicit. In several places in the literature, the definition of coCartesian is instead phrased as requiring every "2-dimensional face of $\mathrm{N}(\mathcal{P}([n]))$ to be a cocartesian square, i.e. a pushout square". To see (at least heuristically) that these two definitions are equivalent use definition A.0.3 inductively on the cardinality of the vertices of $\mathrm{N}(\mathcal{P}([n]))$. A more precise (and slightly stronger) version of this equivalent definition of strongly coCartesian is given by the following result.

**Lemma 1.1.4.** *(Proposition 6.1.1.15. in [21]) Let $\mathcal{C}$ be a category with finite colimits and let $X$ be an $n$-cube. Then the following are equivalent:*

*(i) $X$ is strongly coCartesian.*

*(ii) For every pair of finite sets $T, T' \subset [n]$ the diagram*

$$
\begin{tikzcd}
X(T \cap T') \arrow[r] \arrow[d] & X(T) \arrow[d] \\
X(T') \arrow[r] & X(T \cup T')
\end{tikzcd}
$$

*is a pushout.*

*(iii) For every $T \subset [n]$ and every $s \in [n] \backslash T$ the following diagram*

$$
\begin{tikzcd}
X(\emptyset) \arrow[r] \arrow[d] & X(T) \arrow[d] \\
X(\{s\}) \arrow[r] & X(T \cup \{s\})
\end{tikzcd}
$$

*is a pushout.*

We will not give a more detailed proof of this result than the heuristic presented above.

Now that we have these definitions we can finally define what it means for a functor to be $n$-excisive.

**Definition 1.1.5.** (Definition 6.1.1.3 in [21]) A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be $n$-excisive if it carries strongly coCartesian $n$-cubes to Cartesian $n$-cubes. These functors assemble into a full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denoted by $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$.

Once again it is instructive to study the cases $n = 0, 1$, we follow [21]. A 0-cube is simply a morphism, any 0-cube is vacuously strongly coCartesian and clearly cartesian if and only if the corresponding morphism is a weak equivalence. So a 0-excisive functor $F : \mathcal{C} \to \mathcal{D}$ has to send every morphism of $\mathcal{C}$ to a weak equivalence in $\mathcal{D}$. In other words, a functor $F$ is 0-excisive if and only if it factors through the maximal subgroupoid of $\mathcal{D}$. Notice that this means (if $\mathcal{C}$ is connected), that up to weak equivalence a 0-excisive functor is constant, which is what we expect if it is indeed analogous to degree 0-polynomials.
A 1-cube is just a commutative square, it is cartesian if this square is a pullback square, and it isn't hard to say that a 1-cube is strongly coCartesian if it is a pushout square, which is clear consequence of the definition (A.0.3). So we see that a 1-excisive functor is a functor that sends pushouts to pullbacks.

We now state the following pair, which we combine for ease of reference, of technical results, though only briefly discuss their proofs.

**Proposition 1.1.6.** *(Proposition 6.1.1.13. and 6.1.1.13. in [21]) Let $n$ be an integer and $m \leq n$ a subset. If $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ is a strongly coCartesian $n$-cube, then every $m$ face is also strongly coCartesian.*

*Let $n$ be an integer and $T \subset [n]$ a subset. If every $T$-face of $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ is a Cartesian $T$-cube, then $X$ is a Cartesian $n$-cube.*

For the first proposition, the first challenge is simply dissecting definition, which reduces the proof to computing a certain colimit. This colimit can be computed by changing the diagram category to an equivalent one and using that $X$ is strongly coCartesian. The main idea of the proof is the change of diagram category.
The second proposition is proven with a similar idea, the proof consists in computing a certain limit. The key idea is again the change of diagram category, at which point the result follows clearly from the assumptions.

**Corollary 1.1.7.** *(Corollary 6.1.1.4. in [21]) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between $\infty$-categories. Assume that $\mathcal{C}$ admits finite colimits and finite limits. If $F$ is $m$-excisive then it is $n$-excisive for all $n \geq m$.*

*Proof.* let $F : \mathcal{C} \to \mathcal{D}$ be an $m$-excisive functor, which means it sends strongly coCartesian $m$-cubes to Cartesian $m$-cubes. Now take a strongly coCartesian $n$-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$, for some $n \geq m$. By the first of the above propositions, every $m$-face must be strongly coCartesian as well, and so $F$ sends each $m$-face to a Cartesian $m$-face. In particular, every $m$-face of $FX$ is Cartesian, thus by the second of the above propositions, $FX$ is Cartesian. $\square$

## 1.2 Degree $n$ approximation

We prove the following theorem, which in some sense makes Goodwillie calculus possible.

**Theorem 1.2.1.** *(Theorem 6.1.1.10. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. Then the natural inclusion functors $\mathrm{Exc}^n(\mathcal{C},\mathcal{D}) \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ admits a left adjoint which we denote by $P_n$. Further these functors are left exact (i.e. preserve finite limits).*

We will in fact construct the $P_n$ explicitly, and then prove that these functors have all the desired properties. The construction of $P_n$, will be done in three steps the first of which is defining a functor $\mathcal{C} \times \mathrm{N}\,\mathrm{Fin}^{\mathrm{i}} \to \mathcal{C}$ where $\mathrm{Fin}^{\mathrm{i}}$ is the category of finite sets and injections.

**Definition 1.2.2.** (Construction 6.1.1.8. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object. We consider functors $F \in \mathrm{Fun}(\mathrm{N}(\mathrm{Fin}^{\mathrm{i}}),\mathcal{C})$ satisfying the following two properties:

(i) $F$ maps sets of cardinality 1 to the final object of $\mathcal{C}$,

(ii) For every finite set $S$, $F(S)$ is a colimit of the diagram $F|_{\mathrm{N}(\mathcal{P}_{\leq 1}(S))}$.

Notice that (ii) is exactly equivalent to requiring that $F$ is a left Kan extension (A.0.3) of its restriction to $\mathrm{N}(\mathcal{P}_{\leq 1}(S))$. We will now use proposition A.0.5 in order to obtain a functor $\mathcal{C} \times \mathrm{N}(\mathrm{Fin}^{\mathrm{i}}) \to \mathcal{C}$, where we will denote the image of $(X, S)$ by $C_S(X)$.
Applied directly, taking $\mathrm{N}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1})$ as full subcategory of $\mathrm{N}(\mathrm{Fin}^{\mathrm{i}})$, proposition A.0.5 yields a trivial fibration $\mathcal{K} \to \mathcal{K}'$ from the category of functors $\mathrm{N}(\mathrm{Fin}^{\mathrm{i}}) \to \mathcal{C}$ which are left Kan extensions of their restriction to $\mathrm{N}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1})$ (i.e. satisfying (ii)) to the category of functors $F : \mathrm{N}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1}) \to \mathcal{C}$ such that for every finite $S$ set the induced diagram $\mathrm{N}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1})_{/S}$ admits a colimit. As the diagram category is skeletally finite we see that, because $\mathcal{C}$ admits finite colimits, the condition is vacuously true, so we get a a trivial fibration $\mathcal{K} \to \mathrm{Fun}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1},\mathcal{C})$. Trivial fibrations admit sections, thus giving us a map $s : \mathrm{Fun}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1},\mathcal{C}) \to \mathcal{K}$ and now we may forcefully enforce condition (ii). We can restrict $s$ to those functors mapping sets of cardinality 1 to the terminal object of $\mathcal{C}$. Because $s$ is a section of the restriction functor, we see that $s$ of such a functor must also satisfy property (ii). Finally, observe that if we restrict $\mathrm{Fun}(\mathrm{Fin}^{\mathrm{i}}_{\leq 1},\mathcal{C})$ to those functors satisfying (ii), then this category is equivalent to $\mathcal{C}$ as these functors are uniquely determined by where they send the empty set.
Thus we have a map $s| : \mathcal{C} \to \mathrm{Fun}(\mathrm{Fin}^{\mathrm{i}},\mathcal{C})$, such that the image of any object is a left Kan extension of its restriction to $\mathrm{Fin}^{\mathrm{i}}_{\leq 1}$, which by adjunction gives us the desired functor $\mathcal{C} \times \mathrm{N}(\mathrm{Fin}^{\mathrm{i}}) \to \mathcal{C}$.

By construction the values $C_S(X)$ are forced when $|S| \in \{0,1\}$, thus we already know that $C_\emptyset(X) = X$ and $C_{\{*\}}(X) = *$, where $*$ is some final object of $\mathcal{C}$. Now because $C_\bullet(X)$ is a left Kan extension of its restriction to sets of cardinality less than or equal to 1, we can understand $C_S(X)$ in general as a colimit involving one copy of $X$ and many copies of $*$. In particular for $S$ of cardinality 2, it isn't hard to see that $C_S(X)$ is equivalent to $\Sigma X$.

From the definition and the fact that colimits commute with colimits, we can see that $C_S(-)$ preserves colimits and final objects.

Now we can move on to the second step in our construction.

**Definition 1.2.3.** (Construction 6.1.1.22. in [20]) Let $\mathcal{C}$ be a category which admits finite colimits and a terminal object, let $\mathcal{D}$ be a category which admits finite limits and let $F : \mathcal{C} \to \mathcal{D}$ be a functor between these categories. For each integer $n$, we define $T_n(F) : \mathcal{C} \to \mathcal{D}$ by

$$T_n(F)(X) := \varprojlim_{\emptyset \neq S \subset [n]} F(C_S(X)).$$

We have a natural map $F(X) = F(C_\emptyset(X)) \to T_n(F)(X)$ by the universal property, and these assemble into a natural transformation $\theta_F : F \to T_n F$. Furthermore, it isn't hard to see that this natural transformation depends functorially on $F$.

We will, in time, need the following technical lemma, which we state without detailed proof.

**Lemma 1.2.4.** *(Lemma 6.1.1.26. in [21]) Let $\mathcal{C}$ be an $\infty$-category which admits finite colimits and a final object, $\mathcal{D}$ a category with finite limits and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let $X$ be a strongly coCartesian $n$-cube in $\mathcal{C}$, the canonical map of $n$-cubes $\theta_F : F(X) \to T_n F(X)$ factors through a Cartesian cube of $\mathcal{D}$.*

A strongly coCatersian cube $X$, induces a whole family of cubes $X_I : I' \mapsto X(I \cup I')$. This means we now have a cube at every vertex of our original cube. These "vertex"-cubes $X_I$ are the ones we use to define $Y$, inspired by how the definition of $T_n$ maps the cubes $C_\bullet(X)$ to an object of $\mathcal{C}$, we define

$$Y(I) := \varprojlim_{\emptyset \neq S \subset [n]} F(X_I)(S).$$

In essence we have done to every "vertex"-cubes what $T_n$ does to the cubes $C_\bullet(X)$, and what this construction gives is ann approximation to what $F(\emptyset)$ ought to be if we are hoping for a Cartesian cube. We could have tried to apply this idea directly to the cube $F(X)$, but this turns out not to have enough leeway for our purposes. One may observe that the map $\theta_F : F(X) \to T_n(F(X))$ factors through $Y$, so all that remains to see is that $Y$ is Cartesian. For this, because limits commute with limits, so Cartesian cubes are closed under limits, it suffices to show that the cubes $I \mapsto X_I(S)$ are Cartesian for each non empty $S$. To prove this, we use the technical lemmas of section §1.1.6 and study the $S$-faces of these cubes.

Finally, from here we can define the functors of interest to us.

**Definition 1.2.5.** (Construction 6.1.1.27.) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ be a differentiable category and $F : \mathcal{C} \to \mathcal{D}$ be a functor between these categories. For each integer $n$, denote by $P_n$ the colimit of

$$F \xrightarrow{\theta_F} T_n F \xrightarrow{\theta_{T_n F}} T_n T_n F \xrightarrow{\theta_{T_n T_n F}} T_n T_n T_n F \to \cdots .$$

We of course have a natural transformation $\phi : F \to P_n F$, functorial in $F$.

In due time, we will call this the $n$-excisive approximation of $F$, though for now we don't know many (if any) properties of $P_n$. Our goal main for the remainder of this section is to show that the $P_n$ we constructed satisfies all the properties of theorem 1.2.1. First, in order to perhaps gain some intuition for this result we state the following simple computational result, without proof as the result is more or less immediate.

**Proposition 1.2.6.** *(Example 6.1.1.23. and 6.1.1.28. in [21]) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a category with small colimits and a final object to a differentiable category. Then $P_0 F$ is the constant functor with value $F(*)$ where $*$ is a final object of $\mathcal{C}$.*
*Further if $F$ preserves final objects, we have that $P_1 F \simeq \varinjlim_{m \geq 0} \Omega^m_{\mathcal{D}} \circ F \circ \Sigma^m_{\mathcal{C}}$.*

The fact that these functors preserve finite limits is the easy part of theorem 1.2.1.

**Lemma 1.2.7.** *(Remark 6.1.1.24. and 6.1.1.29. of [21]) The functors $P_n$ constructed above, viewed as endofunctors of* $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$*, are left exact.*

*Proof.* We first notice that as $T_n F$ is defined as a limit, and as limits commute with each other by lemma A.0.6, it isn't hard to see that

$$\varprojlim_{i \in I} T_n F_i \simeq \varprojlim_{i \in I} \varprojlim_{\emptyset \neq S \subset [n]} F_i(C_S(-)) \simeq \varprojlim_{\emptyset \neq S \subset [n]} \varprojlim_{i \in I} F_i(C_S(-)) \simeq T_n \varprojlim_{i \in I} F_i.$$

Now to deduce from this that $P_n$ preserves finite limits, we use that $P_n$ is a sequential colimit of functors which preserve limits. By assumption, sequential colimits in $\mathcal{D}$ are left exact, and so it is clear that $P_n$ preserves finite limits pointwise, which is enough for $P_n$ to preserve finite limits of functors. $\square$

We will in time prove that $P_n$ preserves all colimits, as we will prove that they are left adjoint, however we do need to show by hand that $P_n$ preserves some colimits in order to prove that $P_n$ is a left adjoint. In both the above result and the following, it may seem redundant to specify that we view $P_n$ as an endofunctor of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and strictly speaking this information does indeed not contribute anything. However, it helps with clearing up ideas, as in time we see $P_n$ as a functor whose codomain is those functors which are $n$-excisive.

**Lemma 1.2.8.** *(Remark 6.1.1.31. in [21]) The functors $P_n$, viewed as endofunctors of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, preserve sequential colimits, i.e. colimits of $\mathrm{N}(\mathbb{Z})$ shaped diagrams.*

*Proof.* Sequential colimits in $\mathcal{D}$ commute with finite limits by assumption, so because $T_n$ is constructed by a finite limit we see that

$$\varinjlim_{\mathrm{N}(\mathbb{Z})} T_n F_n \simeq T_n \varinjlim_{\mathrm{N}(\mathbb{Z})} F_n.$$

As $P_n$ is a sequential colimit, and sequential colimits commute with each other by lemma A.0.6, we get the desired result from the result for $T_n$. $\square$

Another relatively simple result is the following. Heuristically, this result gives us a class of functors such that composition by these is to $P_n$ what multiplication by scalars is to ordinary Taylor approximation.

**Lemma 1.2.9.** *(Remark 6.1.1.30. and 6.1.1.32. of [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ a differentiable category, a functor $F : \mathcal{C} \to \mathcal{D}$, $\mathcal{C}'$ another category with finite colimits and a final object and a functor $G : \mathcal{C}' \to \mathcal{C}$ which preserves pushouts and a final object. Then we have*

$$P_n(F \circ G) \simeq P_n(F) \circ G.$$

*Similarly, if $\mathcal{D}'$ is another differentiable category and $H : \mathcal{D} \to \mathcal{D}'$ is a functor which preserves finite limits and sequential colimits, then we have*

$$P_n(H \circ F) \simeq H \circ P_n(F).$$

*Proof.* Using the same notation as in the statement of the result, we first prove that $P_n(F \circ G) \simeq P_n(F) \circ G$. Because $C_\emptyset(X) \simeq X$, it is obvious that $G$ commutes with $C_\emptyset$. Because $G$ preserves final objects, the fact that $C_{\{s\}}(X) = *$, where $* \in \mathcal{C}$ is some final object, we can observe that $G$ commutes with $C_{\{s\}}$. Now because $C_\bullet(X)$ is a left Kan-extension of its restriction to sets of cardinality at most one, we have (by induction using the shape of the colimit diagram) from the fact that $G$ preserves pushouts that

$$G \circ C_S(X) \simeq C_S(GX).$$

From this the desired result is obvious, as $G$ becomes "part of the argument" and is not affected by our various constructions.

The fact that $P_n(H \circ F) \simeq H \circ P_n(F)$ is immediate, as the assumptions on $H$ correspond exactly to the limits used to construct $P_n$. $\square$

*Remark* 1.2.10. Pursuing the analogy with ordinary calculus, we see that functors satisfying the assumptions of the above result can be considered as right (resp. left) scalars. Specializing to a specific example, a natural question becomes what are the left/right/two-sided scalars with respect to Goodwillie calculus of endofunctors of Top. For two sided scalars, we see that the two sided scalars are determined by where they send $S^0$. Indeed, preservation of pushouts and final object means $E(S^n) = \Sigma^n E(S^0)$, and up to homootopy everything is a CW-complex, i.e. a sequential colimit of $n$-skeleta, which are iterated pushouts, and so $E(S^0)$ determines the (weak) homotopy type of $E(X)$. If a functor is just a right scalar or just a left scalar, we weren't able to find any interesting unified description.

We are almost to ready to prove theorem 1.2.1, we just need to prove the next two lemmas. The first of which proves that the codomain of $P_n$ can be chosen to be $n$-excisive functors.

**Lemma 1.2.11.** *(Lemma 6.1.1.33. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ be a differentiable category and $F : \mathcal{C} \to \mathcal{D}$. Then the functor $P_nF : \mathcal{C} \to \mathcal{D}$ is $n$-excisive.*

*Proof.* Let $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$ be a strongly coCartesian cube, we want to show that $P_nF(X)$ is a Cartesian cube of $\mathcal{D}$. We can write $P_nF(X)$ as the colimit of the following sequence of cubes:

$$F(X) \to T_nF(X) \to T_nT_nF(X) \to \cdots .$$

Now by lemma 1.2.4, we can add Cartesian cubes $Y_i$ to the diagram, to produce the following commutative diagram:

$$\begin{tikzcd} F(X) \arrow[rr] \arrow[dr] & & T_nF(X) \arrow[rr] \arrow[dr] & & T_nT_nF(X) \arrow[rr] \arrow[dr] & & \cdots \\ & Y_0 \arrow[ur] & & Y_1 \arrow[ur] & & Y_2 \arrow[ur] & \end{tikzcd}.$$

Next using proposition A.0.7 we see that the colimit of the above diagram can be computed both as the sequential colimit of the top row (yielding $P_n(X)$) or of the subdiagram of $Y_i$. By assumption, each $Y_i$ is cartesian and sequential colimits in the codomain category preserve finite limits, so we may deduce that the sequential colimit is also Cartesian. This proves the claim that $P_n(X)$ is Cartesian. ☐

The next lemma goes towards showing that $P_n$ is a localization functor (which is the method we will use to show that it is a left adjoint).

**Lemma 1.2.12.** *(Lemma 6.1.1.34. and 6.1.1.35. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ be a differentiable category and $F : \mathcal{C} \to \mathcal{D}$. Let $\phi : F \to P_nF$ be the natural transformation, then $P_n(\phi) : P_n(F) \to P_n(P_n(F))$ is an equivalence.*

*Proof.* We first prove that the canonical map $\theta : F \to T_nF$ is sent to an equivalence by $P_n$. Because $P_n$ is left exact by lemma 1.2.7 we have that the natural map

$$P_n(T_nF) = P_n(\varprojlim_{\emptyset \neq S \subset [n]} F \circ C_S) \to \varprojlim_{\emptyset \neq S \subset [n]} P_n(F \circ C_S)$$

is an equivalence. Because $C_S$ preserves pushouts and final objects, we have by lemma 1.2.9 $P_n(F \circ C_S) \simeq P_n(F) \circ C_S$. And so, combining these two equivalences we get an equivalence

$$P_n(T_nF) \to \varprojlim_{\emptyset \neq S \subset [n]} P_n(F) \circ C_S.$$

And so showing that $P_n(\theta) : P_nF \to P_n(T_nF)$ is an equivalence can be done by showing that $P_nF \to \varprojlim_{\emptyset \neq S \subset [n]} P_n(F) \circ C_S$ is an equivalence. Now we have proven in lemma 1.2.11 that $P_nF$ is $n$-excisive, which means that $P_nF$ maps the strongly coCartesian $n$-cube $S \mapsto C_S(X)$, where $S$ runs over the subsets of $[n]$, to a Cartesian $n$-cube. The statement that $S \mapsto P_nF(C_S(X))$ is a Cartesian cube is exactly the statement we are trying to prove, hence we have that $P_n(\theta) : P_nF \to P_n(T_nF)$ is an equivalence.
By composing equivalences we get that the natural map $P_n(\theta^k) : P_nF \to P_n(T_n^kF)$ is an equivalence. Taking the sequential colimit of these maps we get that $P_nF \to \varinjlim_{\mathrm{N}(\mathbb{Z})} P_n(T_n^kF)$ is an equivalence. But $P_n$ preserves sequential colimits by lemma 1.2.8, in such a way that we get an equivalence $P_nF \to P_nP_nF$. Thus concluding the proof. ☐

And we are now ready to prove theorem 1.2.1, which is what all the above constructions and lemmas were leading up to.

*Proof.* All the lemmas of this section almost prove the desired result. In particular we have a left exact functor $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$, which is equipped with a natural map $\phi : Id_{\mathrm{Fun}(\mathcal{C},\mathcal{D})} \to P_n$, all that remains to show is that this functor is a left adjoint to the inclusion. We will do this by using proposition A.0.9, which reduces our work to showing that $P_n$ is essentially surjective, and that

$P_n(\phi_F)$ and $\phi_{P_nF}$ are equivalences for each $F \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.
To show essential surjectivity, let $F : \mathcal{C} \to \mathcal{D}$ be an $n$-excisive functor. This in particular means that $F$ sends, for any $X \in \mathcal{C}$ the strongly coCartesian $n$-cube $C_\bullet(X) : S \mapsto C_S(X)$, where $S$ runs over the subsets of $[n]$, to a Cartesian cube. The statement that $FC_\bullet(X)$ is a Cartesian cube says exactly that the natural map $\theta_F : F \to T_nF$ is an equivalence. As a sequential colimit of equivalences is an equivalence we get that $\phi_F : F \to P_nF$ is an equivalence for any $n$-excisive functor $F$. This shows that $P_n$ is essentially surjective. By lemma 1.2.11 it also shows that $\phi_{P_nF}$ is an equivalence, and we already know that $P_n(\phi_F)$ is an equivalence by lemma 1.2.12, thus concluding the proof. □

From here on out, whenever we write $P_nF$, we implicitly mean that the domain of $F$ admits finite colimits and a final object, whereas the codomain category is a differentiable category.

## 1.3 The difference between successive approximation

In the previous section we constructed functors $P_n : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Exc}^n(\mathcal{C}, \mathcal{D})$ which are left adjoint to the inclusion. Combining the adjunction isomorphism with lemma 1.1.7, we can assemble the $P_m F$ into a tower

$$\cdots \to P_3F \to P_2F \to P_1F \to P_0F$$

which can be intuitively understood as the Taylor tower of $F$. In classical calculus, it is interesting to subtract the degree $n-1$ approximation from the degree $n$ approximation, yielding a monomial which is the homogeneous degree $n$ part of our original function. In this categorical setting this is done by taking the fiber, which we call $D_nF$, of $P_nF \to P_{n-1}F$. We state here the following definitions which give us a way to speak of homogeneous functors, further expanding our dictionary between classical and Goodwillie calculus.

**Definition 1.3.1.** (Def 6.1.2.1. in [21]) Let $\mathcal{C}$ be a category that admits finite colimits and has a final object, $\mathcal{D}$ be a differentiable category and $F : \mathcal{C} \to \mathcal{D}$ a functor. Then by theorem 1.2.1, we have functors $P_nF$. We call $F$ *$n$-reduced* if $P_{n-1}F$ sends any object of $\mathcal{C}$ to a final object of $\mathcal{D}$. And we call $F$ *$n$-homogeneous* if it is $n$-reduced and $n$-excisive.
We assemble the 1-reduced functors into a full subcategory of the functor category denoted by $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ and similarly for the $n$-homogeneous functors, which we denote by $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.

Notice that by the computation that $P_0F$ is the constant functor with value $F(*)$ for some final object $* \in \mathcal{C}$, we see that being 1-reduced can be restated as saying that $F$ preserves the final object.

Understanding this $D_nF$ usually requires specifying the domain or codomain, if not the functor $F$. However, we have the following result, which says that if $F$ is 1-reduced then $D_nF$ can be delooped once, or in other words that the Taylor tower of a functor is a tower of principal fibrations.

**Theorem 1.3.2.** *(Theorem 6.1.2.4. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, $\mathcal{D}$ a differentiable category. Then, for all $n$, we have a diagram in* $\mathrm{Fun}(\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}), \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$*:*

$$\begin{tikzcd} P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[d] \\ K_n \arrow[r] & R_n \end{tikzcd} .$$

*The functors $P_n, P_{n-1}$ are the $n$-excisive approximations of theorem 1.2.1 implicitly postcomposed with the inclusions $\mathrm{Exc}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, and $R_n$ and $K_n$ satisfy the following properties.*

*(i) For every reduced functor $F : \mathcal{C} \to \mathcal{D}$, $K_n(F)$ carries every object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

*(ii) For every reduced functor $F : \mathcal{C} \to \mathcal{D}$ we have that $R_n(F)$ is $n$-homogeneous.*

*(iii) The functor $R_n$ is left exact.*

*(iv) If $F : \mathcal{C} \to \mathcal{D}$ is $(n-1)$-excisive, then $R(F)$ carries every object of $\mathcal{C}$ to a final object of $\mathcal{D}$.*

To deduce from this the result the fact that for reduced $F$ we have that $D_n$ can be delooped follows from point (i), saying that in this case $P_n \to P_{n-1} \to R_n$ is a fibration, and so by looking at the Puppe fiber sequence we have that the fiber of $P_n \to P_{n-1}$ is $\Omega R_n$.

Beyond the philosophical interest in this result, we will prove the above result in order to get the following corollary, which will be useful in proving the main result of §2.2.

**Corollary 1.3.3.** *(Corollary 6.1.2.9. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category. Then composition by $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ induces for any integer $n$ an equivalence*

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}).$$

In order to prove theorem 1.3.2, we will need a more general and more potent version of construction 1.2.2. Fix two integers $n$ and $m$. Consider the full subcategory $\mathcal{P} = \mathcal{P}_{>0}([n])$ of subsets of $[n]$ which aren't empty. We can define a functor

$$\chi_m : \mathrm{N}(\mathcal{P}^m) \times \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}),$$

which maps a functor $F$ and an $m$-tuple $(S_1, ..., S_m)$ to $F \circ C_{S_1} \circ \cdots \circ C_{S_m}$, where $C_{S_i}$ is the construction of definition 1.2.2. We use these to define, for each $I \subset \mathcal{P}^m$, the functor $U_I : \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, which map a functor $F$ to

$$\varprojlim_{(S_1,...,S_m)\in I} \chi_m|_{\mathrm{N}(I)\times \mathrm{Fun}_*(\mathcal{C},\mathcal{D})}(S_1, ..., S_m, F).$$

We will use the above construction, to construct a diagram in $\mathrm{Fun}(\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}), \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$, which we'll use to pointwise create the diagram of interest to us. Before doing this, we make three preliminary remarks. First, fix $I \subset \mathcal{P}^m, I' \subset \mathcal{P}^n$, and notice that we can naturally view $I \times I'$ as a subset of $\mathcal{P}^{m+n}$. Using that the nerve preserves products and that limits commute with each other by (the dual of) lemma A.0.6, one can readily deduce that $U_{I\times I'} \simeq U_{I'} \circ U_I$. Second, the functors $U_I$ are all left exact by the dual of lemma A.0.6, so that the diagram will in fact be a diagram of left exact endofunctors of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$. Third, notice that the association $I \mapsto U_I$, can naturally be extended to a contravariant functor $\mathcal{P}^m \to \mathrm{Fun}(\mathrm{Fun}_*(\mathcal{C}, \mathcal{D}), \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$.

Now, we need to find an interesting diagram in $\mathcal{P}^m$, however disappointingly, the interest of the diagram will only become clear once we successfully use it to prove theorem 1.3.2. For this, let $B \subset \mathcal{P}$ consists of those subsets of $[n]$ which have non-empty intersection with $[n-1]$. Also, let $A_m \subset \mathcal{P}^m$ consist of those tuples, such that at least one subset contains $n$. We have the following commutative diagram of subsets of $\mathcal{P}^{m+1}$:

$$\begin{tikzcd}
\mathcal{P}^{m+1} & B^m \times \mathcal{P} \arrow[l] & B^{m+1} \arrow[l] \\
A_{m+1} \arrow[u] & A_{m+1} \cap (B^m \times \mathcal{P}) \arrow[l] \arrow[u] & A_{m+1} \cap (B^{m+1}) \arrow[l] \arrow[u] \\
A_m \times \mathcal{P} \arrow[u] & (A_m \cap B^m) \times \mathcal{P} \arrow[l] \arrow[u] &
\end{tikzcd}.$$

By the preceeding discussion, we can turn this into a corresponding diagram of left exact endofunctors of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$, which for future reference we call $\tau_m$:

$$\begin{tikzcd}
U_{\mathcal{P}^{m+1}} \arrow[r] \arrow[d] & U_{B^m\times\mathcal{P}} \arrow[r] \arrow[d] & U_{B^{m+1}} \arrow[d] \\
U_{A_{m+1}} \arrow[r] \arrow[d] & U_{A_{m+1}\cap(B^m\times\mathcal{P})} \arrow[r] \arrow[d] & U_{A_{m+1}\cap(B^{m+1})} \\
U_{A_m\times\mathcal{P}} \arrow[r] & U_{(A_m\cap B^m)\times\mathcal{P}} &
\end{tikzcd}.$$

We now prove two lemmas to help us understand the above diagram, and with those in hand we will move on to the proof of theorem 1.3.2.

**Lemma 1.3.4.** *(Lemma 6.1.2.13. in [21]) For each $m \geq 0$, the functors $U_{A_m}$ takes every $F \in$* $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ *to a final object of the functor category. In particular, $U_{A_m}$ is a final object in the category of endofunctors of* $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$.

*Proof.* The goal is to show that for each reduced functor $F : \mathcal{C} \to \mathcal{D}$, and for every object $X \in \mathrm{Ob}(\mathcal{C})$, the limit $\varprojlim_{(S_1,...,S_m)\in A_m}(F \circ C_{S_1} \circ \cdots \circ C_{S_m})$ maps $X$ to a final object. We will do this by finding a

different diagram over which to compute this limit. Let $A'_m = \mathcal{P}^m \backslash B^m$, which naturally includes into $A_m$. Notice that an $m$-tuple in $A'_m$ must have at least one, say $S_i$ term which is the singleton $\{n\}$, which by definition of $C_{S_i}$ and reducedness of $F$ implies that the over $A'_m$ the limit of interest is as desired. So it suffices to show that the map $\mathrm{N}(A'_m) \to N(A_m)$ is initial.
By theorem A.0.10, it will suffice to show, for each $(S_1, ..., S_m) \in A_m$, the nerve of the following category is contractible:

$$W = \{(S'_1, ..., S'_m) \in A'_m | S'_i \subset S_i, \forall 1 \le i \le m\}.$$

Let $I \subset \{1, ..., n\}$ be the subset of indices $i$ such that $n \in S_i$, which is non empty by definition of $A_m$. Now for each $\emptyset \neq J \subset I$, let $W_J$ denote the subcategory of $W$ consisting of tuples for which $S'_j = \{n\}, \forall j \in J$.Now it is quite clear that $N(W_J)$ is contractible for each $J$, as each $W_J$ has a terminal object given by $(S'_1, ..., S'_m)$ with $S'_i = \{n\}$ if $i \in J$ and $S'_i = S_i$ otherwise. We claim that this implies that $W$ has contractible nerve.
For this, observe that $N(W)$ is the homotopy colimit of the contractible $N(W_J)$. In general, a homotopy colimit of contractible spaces isn't contractible, however here the diagram category is contractible. This allows us to conclude by corollary 29.2 of [5]. ☐

**Lemma 1.3.5.** *(Lemma 6.1.2.14. in [21]) The maps $U_{A_{m+1}} \to U_{A_m \times \mathcal{P}}$ and $U_{A_{m+1} \cap (B^m \times \mathcal{P})} \to U_{(A_m \cap B^m) \times \mathcal{P}}$ are natural equivalences.*

*Proof.* We will only prove the result for the map $U_{A_{m+1} \cap (B^m \times \mathcal{P})} \to U_{(A_m \cap B^m) \times \mathcal{P}}$, the case of the other map being wholly similar and done in [21]. The only idea is to use the dual of proposition A.0.8 and the decomposition of $N(A_{m+1})$ as $N(A_m \times \mathcal{P}) \cup N(B^m \times A_1)$. Indeed, applying the dual of proposition A.0.8, to the diagram defining $U_{A_{m+1} \cap (B^m \times \mathcal{P})}$ with the decomposition just described, we get the following pullback square

$$\begin{tikzcd}
U_{A_{m+1} \cap (B^m \times \mathcal{P})} \arrow[r] \arrow[d] & U_{(B^m \times A_1) \cap (B^m \times \mathcal{P})} \arrow[d] \\
U_{(A_m \times \mathcal{P}) \cap (B^m \times \mathcal{P})} \arrow[r] & U_{((B^m \cap A_m) \times A_1) \cap (B^m \times \mathcal{P})}
\end{tikzcd}.$$

Now, using $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$ and the formula $U_{I \times I'} \simeq U_{I'} \circ U_I$ which has been discussed previously, it is easy to see that this diagram is equivalent to:

$$\begin{tikzcd}
U_{A_{m+1} \cap (B^m \times \mathcal{P})} \arrow[r] \arrow[d] & U_{A_1} \circ U_{B^m} \arrow[d] \\
U_{((A_m \cap B^m) \times \mathcal{P})} \arrow[r] & U_{A_1} \circ U_{B^m \cap A_m}
\end{tikzcd}.$$

Now, by the previous lemma, the two functors on the right map every functor a final object, thus are themselves final object in the appropriate category. In particular the right-vertical map is an equivalence. And because equivalences are stable under pullback, the left vertical map is an equivalence, and this what we wanted to show. ☐

We are now ready to prove theorem 1.3.2

*Proof.* We first recall to the reader that when we write $\tau_m$, we mean the diagram defined above. We define another diagram, $\sigma_m$, also in the category of endofunctors of $\mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$:

$$\begin{tikzcd}
U_{\mathcal{P}^m} \arrow[r] \arrow[d] & U_{B^m} \arrow[d] \\
U_{A_m} \arrow[r] & U_{B^m \cap A_m}
\end{tikzcd}.$$

Noticing that the functor $T_n$ which we constructed in the previous section can be described as $U_{\mathcal{P}}$. Using the second of the above two lemmas and the formula $U_{I\times I'} \simeq U_{I'} \circ U_I$, we can identify the diagram $T_n\sigma_m$ with

$$\begin{array}{ccc} U_{\mathcal{P}^{m+1}} & \longrightarrow & U_{B^m\times\mathcal{P}} \\ \downarrow & & \downarrow \\ U_{A_{m+1}} & \longrightarrow & U_{A_{m+1}\cap(B^m\times\mathcal{P})} \end{array} .$$

Referring to $\tau_m$, we see we have a natural transformation $\alpha_m : T_n(\sigma_m) \to \sigma_{m+1}$, and we, by construction of $T_n$, have a canonical map $\sigma_m \to T_n(\sigma_m)$. We can stick these transformations together, to obtain

$$\sigma_0 \to T_n(\sigma_0) \to \sigma_1 \to T_n(\sigma_1) \to \cdots .$$

Taking the colimit of the above diagram, which exists because sequential colimits exist in $\mathcal{D}$ by assumption, gives us another diagram, which we will call $\sigma_\infty$, which will turn out to be the desired diagram. To fix notation, explicitly write $\sigma_\infty$ as

$$\begin{array}{ccc} P & \longrightarrow & P' \\ \downarrow & & \downarrow \\ K & \longrightarrow & R \end{array} .$$

By the formula for $U_{I\times I'}$, we can identify $U_{\mathcal{P}^m} \simeq T_n^m$, which immediately implies that $P$ is $P_n$.
Next, we need to understand $U_B$, which is done using the evident inclusion $\mathcal{P}([n-1]) \to B$. Because this map clearly preserves supremums, we have that it must be a left adjoint by the adjoint functor theorem for posets, in particular it is an initial map by corollary 7.2.3.7. of [22]. Thus $U_B \simeq U_{\mathcal{P}([n-1])}$, which we already know to be equivalent to $T_{n-1}$, and so we may repeat the same reasoning used to identify $P$, to see that $P' \simeq P_{n-1}$.
The functor $K$ is the colimit of the functors $U_{A_m}$, which all map every reduced functor to a functor which maps everything to a final object by the first of the above two lemmas, so the same is true of $K$.
Now, we show that $R$ preserves finite limits, i.e. is left exact. Because limits commute with limits by the dual of lemma A.0.6, and because $U_I$ is defined via a limit (and precomposition, which preserves limits), we may deduce that $U_I$ preserves limits. Now because in $\mathcal{D}$ sequential colimits commute with finite limits, this implies that $R$ preserves finite limits, as a sequential colimit of functors which preserve limits.
We now show that $\sigma_\infty$ is a pullback square. Again, because in $\mathcal{D}$ sequential colimits preserve finite limits, it will suffice to show that each $\sigma_m$ is a pullback, which follow from proposition A.0.8 using that $\mathrm{N}(\mathcal{P}^m) = \mathrm{N}(B^m) \cup \mathrm{N}(A_m)$, with intersection $\mathrm{N}(B^m \cap A_m)$.
We now prove that if $F$ is $(n-1)$-excisive in addition to being reduced, we have that $R(F)$ is a final object of $\mathrm{Fun}_*(\mathcal{C},\mathcal{D})$. Because $R$ is the colimit of $U_{B^m\cap A_m}$, it will suffice to show that the claim holds for each of these. The subcategory $B^m \cap A_m$ can be seen to be in bijection with $\mathcal{P}_{>0}([n-1]) \times \mathcal{P}_{>0}(\{1,...,m\})$, by mapping an $m$-tuple $(S_1,...,S_m)$ in the intersection to

$$((S_1 \cap [n-1], ..., S_m \cap n-1), \{i | n \in S_i\}).$$

Using this, we can describe $U_I$, as the limit of a functor $G : \mathrm{N}(\mathcal{P}_{>0}([n-1]))^m \times \mathrm{N}(\mathcal{P}_{>0}(\{1,...,m\})) \to \mathrm{Fun}_*(\mathcal{C},\mathcal{D})$, which by commutativity of limits amongst themselves (i.e. by the dual of A.0.6), means we can compute $U_{B^m\cap A_m}$ as the following iterated limit:

$$\varprojlim_{\emptyset\neq T\subset\{1,...,m\}} \varprojlim_{\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m} G|_{\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m\times\{T\}}.$$

As a limit of final objects is final, it suffices to show that $\varprojlim_{\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m} G|_{\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m\times\{T\}}$ is a final object. For simplicity, write $G_T := G|_{\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m\times\{T\}}$. Making what we want to compute slightly more explicit, we have

$$\varprojlim_{(S_1,...,S_m)\in\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m} G_T(F) = \varprojlim_{(S_1,...,S_m)\in\mathrm{N}(\mathcal{P}_{>0}([n-1]))^m} F \circ C_{S_1'} \circ \cdots C_{S_m'},$$

where $S'_i = S_i$ when $i \notin T$ and $S'_i = S_i \cup \{n\}$ when $i \in T$. Now we may view this limit, as an interation of $m$-limits, each of which is an $(n-1)$-cube, which is strongly coCartesian, so that we may compute the above expression as $F \circ C_{S_1} \circ \cdots \circ C_{S_m}$, with $S_i$ the empty set if $i \notin t$ and $\{n\}$ otherwise. Recalling that $C_{\{n\}}$ maps every object to a terminal object, that $C_{\emptyset}$ is the identity, and using that $T$ is non-empty, we see that the composition $C_{S_1} \circ \cdots \circ C_{S_m}$ maps every object to a final object independent of $T$. Because $F$ is reduced, this shows that the above limit is indeed a functor which is constantly a final object. Which proves that $R(F)$ is a final functor, whenever $F$ is $(n-1)$-excisive in addition to being reduced.
The final thing to check is that $R(F)$ is always $n$-homogeneous. Because $P_n$ is left exact and a left adjoint so preserves all colimits, we have by construction that $P_{n-1}R(F) \simeq R(P_{n-1}F)$, as $P_{n.1}(F)$ is $(n-1)$-excisive by construction, by what we just proved we have that $R(P_{n-1}F)$ is a final functor, in particular $R(F)$ is $n$-reduced. For $n$-excisiveness, let $X$ be a strongly coCartesian $n$-cube in $\mathcal{C}$, we want to show that $R(F)(X)$ is Cartesian. By definition of $R$, we can compute this cube as the following colimit

$$U_{B^0 \cap A_0}(F)(X) \to T_n(U_{B^0 \cap A_0}(F))(X) \to U_{B^1 \cap A_1}(F)(X) \to T_n(U_{B^1 \cap A_1}(F))(X) \to \cdots .$$

Now by lemma 1.2.4, we can factor the maps $U_{B^m \cap A_m}(F)(X) \to T_n(U_{B^m \cap A_m}(F))(X)$ through Cartesian squares, and now by lemma A.0.7, we obtain that the above colimit is Cartesian. This is what we wanted to show, thus concluding the proof of this result. □

Now that we have completed this proof, we will deduce a series of consequences of this result, culminating with corollary 1.3.3. The first of these result is interesting enough to warrant the title of theorem, rather than just corollary. In order to state this result, recall that $\Lambda^2_2 \subset \Delta^2$ is the $\infty$-category analogue to the diagram category $\bullet \to \bullet \leftarrow \bullet$ .

**Theorem 1.3.6.** *(Theorem 6.1.2.5. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a differentiable category and let $n \geq 1$. Let $\mathcal{E} \subset \mathrm{Fun}(\Lambda^2_2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$ be the full subcategory spanned by those diagrams $E \to H \leftarrow H_0$, where $E$ is reduced and $(n-1)$-excisive, $H$ is $n$-homogeneous and $H_0$ is a final object in the functor category.*
*Then, the pullback map $\varprojlim : \mathrm{Fun}(\Lambda^2_2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ induces an equivalence $\mathcal{E} \to \mathrm{Exc}^n_*(\mathcal{C}, \mathcal{D})$.*

To prove the above result, we will need the following technical result, which we include without proof. One can consider this result as a strengthening of Fubini thanks to the extra assumptions.

**Lemma 1.3.7.** *(6.1.2.6. in [21]) Let $\mathcal{C}$ be a category with finite limits. Suppose we are given a diagram*

$$\begin{array}{ccccc} X_{00} & \longrightarrow & X_{01} & \longleftarrow & X_{02} \\ \downarrow & & \downarrow & & \downarrow \\ X_{10} & \longrightarrow & X_{11} & \longleftarrow & X_{12} \\ \uparrow & & \uparrow & & \uparrow \\ X_{20} & \longrightarrow & X_{21} & \longleftarrow & X_{22} \end{array},$$

*where the maps whose domain is top right or bottom left object are equivalences. Then, denoting by $X^h_i$ the fiber product $X_{i0} \times_{X_{i1}} X_{i2}$ and by $X^v_i$ the fiber product $X_{0i} \times_{X_{1i}} X_{2i}$. Then the diagrams*

$$X^h_0 \to X^h_1 \leftarrow X^h_2$$

*and*

$$X^v_0 \to X^v_1 \leftarrow X^v_2$$

*are pointwise equivalent. Moreover, the equivalence can be chosen to be functorial with respect to the diagram.*

With this in hand, we move on to the proof of the statement.

*Proof.* Because $n$-excisiveness and reducedness are stable under limits, it is clear that we have a map $\phi : \mathcal{E} \to \mathrm{Exc}^n_*(\mathcal{C}, \mathcal{D})$, given by taking fiber products. A map $\psi : \mathrm{Exc}^n_*(\mathcal{C}, \mathcal{D}) \to \mathcal{E}$ the other way is provided by theorem 1.3.2, by mapping a functor $F$ to $P_{n-1}(F) \to R(F) \leftarrow K(F)$. The theorem providing the map $\psi$ also shows that $\phi \circ \psi$ is equivalent to the identity. So all that remains to show is that the opposite composition is also the identity, which we do by using the above lemma.
Consider $E \to H \leftarrow H_0$, with $E$ $(n-1)$-excisive, $H$ $n$-homogeneous and $H_0$ carrying every object of $\mathcal{C}$ to a final object. Denote the fiber product of this diagram by $F$. We want to somehow create a three by three grid as in the statement of the previous lemma, such that taking fiber products in one direction yields $E \to H \leftarrow H_0$ and in the other direction we get $P_{n-1}(F) \to R(F) \leftarrow K(F)$.
For this, consider the following diagram

$$\begin{tikzcd}
P_{n-1}(E) \arrow[r] \arrow[d] & P_{n-1}(H) \arrow[d] & P_{n-1}(H_0) \arrow[l] \arrow[d] \\
R(E) \arrow[r] & R(H) & R(H_0) \arrow[l] \\
K(E) \arrow[r] \arrow[u] & K(H) \arrow[u] & K(H_0) \arrow[l] \arrow[u]
\end{tikzcd} .$$

The entire bottom row consists of functors mapping every object to a final object, by theorem 1.3.2. By left exactness of $P_{n-1}$ and $R$, we get that $P_{n-1}(H_0)$ and $R(H_0)$, these functors also map every object to a final object. Because $H$ is $n$-homogeneous, we have by definition that $P_{n-1}(H)$ maps every object to a final object. And finally, for $R(E)$ every object is sent to a final object by theorem 1.3.2. So the assumptions of the lemma hold, thus all that remains is to understand the vertical and horizontal fiber products.
Taking horizontal fiber products, the bottom evidently becomes a functor mapping every object to a final object, which can alternatively be described as $K(F)$. The middle row and top row are sent to $R(F)$ and $P_{n-1}(F)$ by let-exactness of $R$ and $P_{n-1}$. In total, taking fiber products in the horizontal direction gives us $P_{n-1}(F) \to R(F) \leftarrow K(F)$.
Now in the vertical direction, by theorem 1.3.2, we get the diagram $P_n(E) \to P_n(H) \leftarrow P_n(H_0)$, which since these are all $n$-excisive functors, is equivalent to $E \to H \leftarrow H_0$. This completes the proof by the above lemma. □

Now the above leads to a whole slew of corollaries, which we here inspect. First, notice that the pullback of a diagram $E \to H \leftarrow H_0$ as above, is $n$-homogeneous if and only if $E$ is a final object. This follows from left exactness of $P_{n-1}$. Thus the above result specializes to the following.

**Corollary 1.3.8.** *(Corollary 6.1.2.7. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a differentiable category and let $n \geq 1$. Let $\mathcal{E}_0 \subset \mathrm{Fun}(\Lambda^2_2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D}))$ be the full subcategory spanned by those diagrams $E \to H \leftarrow H_0$, where $E$ is a final object in the functor category, $H$ is $n$-homogeneous and $H_0$ is a final object in the functor category.*
*Then, the pullback map $\varprojlim : \mathrm{Fun}(\Lambda^2_2, \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}_*(\mathcal{C}, \mathcal{D})$ induces an equivalence $\mathcal{E}_0 \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$.*

We specialize this result even further by considering the case where $\mathcal{D}$ is pointed, i.e. the final objects are also initial. In this case, The category $\mathcal{E}_0$ can be identified with $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$, by sending a diagram $E \to H \leftarrow H_0$ as in the statement of the previous result to $H$. Formally, this identification is given by restricting along the inclusion as the middle vertex $\bullet \to (\bullet \to \bullet \leftarrow \bullet)$, which can be observed to be a trivial Kan fibration between the desired categories via lemma A.0.5. Under this identification, it is clear that the map $\mathcal{E}_0 \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ corresponds to the loop space functor $\Omega : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$. Further, the statement that the functor of the previous result is an equivalence becomes the following, quite amazing, statement.

**Corollary 1.3.9.** *(Corollary 6.1.2.8. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category and let $n \geq 1$. Then the infinity category $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is stable.*

From this, we wish to deduce the desired lemma 1.3.3. Recall that this means we want to show that the map $\Omega^\infty_* : \mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$ is an equivalence. A priori the codomain of this map is simply $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, in order to corestrict appropriately, recall that $\Omega^\infty$ is exact, so commutes with $P_n$ by lemma 1.2.9, in particular $n$-homogeneity is preserved by $\Omega^\infty$. In fact, more generally, evaluation $e_K : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ at any finite space $K$ preserves $n$-homogeneity. And as the (co)limits defining $P_n$ can all be computed pointwise, we see that if $e_K \circ F$ is $n$-homogeneous for each finite space $K$, then $F$ itself must have been $n$-homogeneous.
Now restricting and corestricting the currying isomorphism appropriately, we get that

$$\mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Sp}(\mathrm{Homog}^n(\mathcal{C}, \mathcal{D}))$$

which comes equipped with a natural map to $\mathrm{Homog}^n(\mathcal{C}, \mathcal{D})$, which is an equivalence by the previous proposition and proposition A.0.12. Which up to a quick dissection of the definitions proves the desired result.

# 2 Defining the derivative

## 2.1 Multivariable preliminaries

The material we developed in section §1, in analogy with ordinary calculus, admits a multivariable generalization, which we discuss in this section. We study this with the main goal of defining the cross-effect functor, and so we will not cover as much material as what is contained in the corresponding sections of [21].

Unsurprisingly the first thing we will generalize is the notion of $n$-excisive, and theorem 1.2.1.

**Definition 2.1.1.** (6.1.3.1. in [21]) Suppose we are given $\infty$-categories $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ which admit pushouts and $\mathcal{D}$ a category with finite limits. Given a sequence of integers $\vec{n} = (n_1, ..., n_m), n_i \in \mathbb{N}$, we call a functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ $\vec{n}$-excisive if the composition

$$\mathcal{C}_j \to \mathcal{C}_j \times \prod_{i \neq j, i=1}^{m} \{X_j\} \to \prod_{i=1}^{m} \mathcal{C}_i \xrightarrow{F} \mathcal{D}$$

is $n_j$-excisive for any choice of $j$ and objects $\{X_i\}_{i \neq j}$ with $X_i \in \mathcal{C}_i$. In simpler terms, we call $F$ $\vec{n}$-excisive if it is $n_i$-excisive in its $i$th variable.
We denote by $\mathrm{Exc}^{\vec{n}}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$ the collection of all $\vec{n}$-excisive functors. In the special case where $\vec{n} = (1, ..., 1)$ we will allow ourselves to use the superscript $e$.

Notice that a priori studying excisiveness in each variable separately need not relate nicely to excisiveness when seen as a single variable functor out of $\mathcal{C} := \prod_{i=1}^m \mathcal{C}_i$. But, similar to results in classical analysis relating differentiability in each variable separately to full differentiability, we have results relating the above notion of $\vec{n}$-excisive with the notion of $n$-excisive described in subsection §1.1. We cover one such result at the end of this section. To start our analysis of multivariable Goodwillie calculus, we prove the following result, which gives us a multivariable analogue of theorem 1.2.1.

**Proposition 2.1.2.** *(6.1.3.6. in [21]) Let $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ be categories which admit finite colimits and a final object and let $\mathcal{D}$ be a differentiable category. Then, for any sequence of integers $\vec{n}$, the inclusion $\mathrm{Exc}^{\vec{n}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ admits a left adjoint $P_{\vec{n}}$, which is left exact.*

*Proof.* We first notice that the isomorphism, which is just the currying adjunction for simplicial sets.

$$\mathrm{Fun}(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}_1, \mathrm{Fun}(\prod_{i=2}^{m} \mathcal{C}_i, \mathcal{D}))$$

nicely restricts to an isomorphism

$$\mathrm{Exc}^{\vec{n}}(\prod_{i=1}^{m} \mathcal{C}_i, \mathcal{D}) \simeq \mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\prod_{i=2}^{m} \mathcal{C}_i, \mathcal{D}))$$

where $\vec{n}'$ is obtained from $\vec{n}$ by omitting the first integer. This opens the way for a proof by induction on $m$, as one can observe that the inclusion $\mathrm{Exc}^{\vec{n}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ factors as

$$\mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Exc}^{\vec{n}'}(\prod_{i=2}^{m} \mathcal{C}_i, \mathcal{D})) \xrightarrow{\iota'} \mathrm{Exc}^{n_1}(\mathcal{C}_1, \mathrm{Fun}(\prod_{i=2}^{m} \mathcal{C}_i, \mathcal{D})) \xrightarrow{\iota''} \mathrm{Fun}(\mathcal{C}_1, \mathrm{Fun}(\prod_{i=2}^{m} \mathcal{C}_i, \mathcal{D})).$$

To initialize the induction, notice that if $m = 0$ there is nothing to prove. Now assume that the result holds when we have $m - 1$ categories, in particular we have a left adjoint $P_{\vec{n}'} : \mathrm{Fun}(\prod_{i=2}^m \mathcal{C}_i, \mathcal{D}) \to Exc^{\vec{n}'}(\prod_{i=2}^m \mathcal{C}_i, \mathcal{D})$, which is also left exact. Post composition by this functor gives a left adjoint $\mathrm{Exc}^{n_1}(C_1, P_{\vec{n}'})$ to the inclusion $\iota'$, and it isn't hard to observe that this left adjoint is also left exact. Because $\mathcal{D}$ is differentiable and (co)limits of functors are constructed pointwise, we can see that $\mathrm{Fun}(\prod_{i=2}^m \mathcal{C}_i, \mathcal{D})$ is differentiable, so that we can apply theorem 1.2.1 to obtain a left exact left adjoint to $\iota''$. Composing the left adjoint of $\iota'$ and $\iota''$ we obtain the desired left adjoint. $\square$

We will now introduce the generalizations of the notions of homogeneous and reduced to this multivariable context, so that we have access to all the vocabulary we developed in section §1.

**Definition 2.1.3.** (Definition 6.1.3.7. in [21]) Suppose we are given $\infty$-categories $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_m$ which admit finite colimits and final objects and $\mathcal{D}$ a differentiable category. Given a sequence of integers $\vec{n} = (n_1, ..., n_m), n_i \in \mathbb{N}$, we call a functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ *$\vec{n}$-reduced* if the composition

$$\mathcal{C}_j \to \mathcal{C}_j \times \prod_{i \neq j, i=1}^{m} \{X_j\} \to \prod_{i=1}^{m} \mathcal{C}_i \xrightarrow{F} \mathcal{D}$$

is $n_j$-reduced for any choice of $j$ and objects $\{X_i\}_{i \neq j}$ with $X_i \in \mathcal{C}_i$. In simpler terms, we call $F$ $\vec{n}$-reduced if it is $n_i$-reduced in its $i$th variable. We call a functor simply *reduced* if it is $(1, ..., 1)$-reduced. The collection of reduced functors assemble into a category which we denote by $\mathrm{Fun}_*(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$.
Similar to the single variable case, we call a functor *$\vec{n}$-homogeneous* if it is $\vec{n}$-reduced and $\vec{n}$-excisive. We assemble these into a full subcategory of the functor category $\mathrm{Homog}^{\vec{n}}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$. We call $(1, ...1)$-homogeneous functors multilinear, and we denote the category of these functors by $\mathrm{Exc}_*(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$.

Note that in this notation, writing the domain category as a product is important to underline the multivariable nature of these notions. And so changing how we write $\prod_{i=1}^m \mathcal{C}_i$, for example by coupling terms together, changes the meaning of $\mathrm{Fun}_*(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$ (and any of the other functor categories we introduced).
We already observed that for a single variable functor $F : \mathcal{C} \to \mathcal{D}$, calling it reduced means that it preserves final objects. For a multivariable functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ the notion of being reduced can be rephrased as requiring $F$ to map $(X_1, ..., X_m)$ to the final object of $\mathcal{D}$ if even one of the $X_i$ is a final object of $\mathcal{C}_i$.

Now in order to understand these multivariable notions, it is interesting to see what happens when we set $\mathcal{C} := \prod_{i=1}^m \mathcal{C}_i$ and observe how the multivariable properties appear when we view a functor $F : \mathcal{C} \to \mathcal{D}$ has having a single variable. These questions are the functor calculus parallel of understanding how the Jacobian is related to directional derivatives in ordinary multivariable analysis. In our situation this is accomplished by the two following results.

**Proposition 2.1.4.** *(Proposition 6.1.3.4. in [21]) Let $\{\mathcal{C}_i\}_{i=1}^m$ be categories with finite colimits and a final object, let $\{n_i\}_{i=1}^m$ be a sequence of positive integers, let $\mathcal{D}$ be a category with finite limits and let $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ be a functor which is $n_i$ excisive in its $i$th variable. Then viewed as a single variable functor, it is $n$-excisive, where $n = n_1 + ... + n_m$.*

*Proof.* When we view the functor as having one variable, we will write its domain as $\mathcal{C}$, otherwise we will use the notation of the result. Given a strongly coCartesian $n$-cube $X : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}$, we want to show that $F(X)$ is Cartesian. By the universal property of the product and because colimits are computed component wise we have corresponding strongly coCartesian $n$-cubes $X_i : \mathrm{N}(\mathcal{P}([n])) \to \mathcal{C}_i$ such that we have a factorization of $X$ as

$$\mathrm{N}(\mathcal{P}([n])) \xrightarrow{\Delta} \mathrm{N}(\mathcal{P}([n]))^m \xrightarrow{\prod_{i=1}^m X_i} \prod_{i=1}^{m} \mathcal{C}_i.$$

Recall that the nerve commutes with products, so that we can replace the middle term by $\mathrm{N}(\mathcal{P}([n])^m)$. What we want to show can be rephrased as requiring that $Y = F \circ (\prod_{i=1}^m X_i)$ exhibits $Y(\emptyset, \emptyset, ..., \emptyset)$ as the limit of $Y|_{\mathrm{N}(A)}$, where $A \subset \mathcal{P}([n])^m$ is the full subcategory of $m$-tuples $(S, S, ..., S)$ such that $S$ is non-empty.
To compute this colimit, we will use the usual idea of changing the diagram category to something more manageable. In this case, let $B \subset \mathcal{P}([n])^m$ be the full subcategory spanned by those $m$-tuples whose intersection is non-empty. We obviously have an inclusion $\iota : A \to B$. By the adjoint functor theorem for posets we easily see that $\iota$ must be a left adjoint, thus must be initial by corollary 7.2.3.7. in [22] (see the comment in §0.0.8 for a comment on nomenclature), i.e. the natural map $\varprojlim(Y|_{\mathrm{N}(B)}) \to \varprojlim(Y|_{\mathrm{N}(A)})$ is an equivalence.

To compute this limit $\varprojlim(Y|_{\mathrm{N}(B)})$, we will actually prove the stronger statement that $Y$ is a right Kan extension of $Y|_{\mathrm{N}(B)}$. We will do this by dividing the problem. Consider a chain of subcategories

$$B = B_0 \subset B_1 \subset ... \subset B_k = P([n])^m,$$

with the following two properties:

(i) Each $B_j$ is closed upwards, i.e. if we have $(S_1, ..., S_m) \in B_j$ and $(S_1, ..., S_m) \leq (S'_1, ..., S'_m)$, then $(S'_1, ..., S'_m) \in B_j$.

(ii) Each $B_j$ is obtained from $B_{j-1}$ by adding a single element.

What this accomplishes is that it now suffices to show that $Y|_{\mathrm{N}(B_j)}$ is a right Kan extension of $Y|_{N(B_{j-1})}$. Let $S = (S_1, ..., S_m)$ be the unique element added to $B_{j-1}$ to get to $B_j$. This element must belong to the complement of $B_{j-1}$, in particular of $B$, so that the intersection of the $\{S_i\}_{i=1}^m$ is empty. This in turn means the union $\bigcup_{i=1}^m S_i^c = [n]$, now because the cardinality of $[n]$ is $n+1 \geq n_1 + ... + n_m$, there must be at least one $S_i$, say $S_j$, such that $|S_j^c| > n_j$.
Now notice that for $Y|_{N(B_j)}$ to be a right Kan extension of the restriction of $Y$ to $N(B_{j-1})$ it suffices to verify that

$$Y(S) \simeq \varprojlim_{S \to Z \in S/N(B_{j-1})} Y(Z).$$

Notice that the diagram of the above limit is a cube, indeed the $Z$ such that there exists a morphism $S \to Z$ correspond to $m$-tuples $(S'_1, ..., S'_m)$ such that $S_i \subset S'_i$, which in term correspond to $m$-tuples of subsets of $S_i^c$, finally taking a disjoint union, we see that the shape of the diagram is $\mathcal{P}(\bigsqcup_{i=1}^m S_i^c)$. Under this correspondence, the above equivalence becomes $\tilde{Y} : S/N(B_{j-1}) \to \mathcal{D}$, which is Cartesian.
Let $T \subset S_j$ be of cardinality $n_j + 1$, and view this as a subset of $\bigsqcup_{i=1}^m S_i^c$. Then by lemma 1.1.6, it suffices to verify that each $T$-face of $\tilde{Y}$ is Cartesian. It isn't hard to verify this using that $\tilde{Y}$ can be written as $F \circ \tilde{X}$ such that this $\tilde{X}$ is strongly coCartesian (so its $T$-faces as well by lemma 1.1.6), $F$ is $n_j$-excisive, that $T$-faces are $n_j$-cubes and the fact that the $T$ faces are constant on all coordinates different than the $j$th. □

**Proposition 2.1.5.** *(Proposition 6.1.3.10. in [21]) Let $\{\mathcal{C}_i\}_{i=1}^m$ be categories with finite colimits and a final object, let $\mathcal{D}$ be a differentiable category and let $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ be a functor which is* 1*-reduced each variable. Then viewed as a single variable functor, it is $m$-reduced.*

*Proof.* Denote by $F'$ the functor $F$ seen as a one variable functor, by definition what we want to show is that $P_{m-1}F'(X)$ is final for all $X \in \mathrm{Ob}(\mathcal{C})$. Using the same ideas as in the proof of lemma 1.2.11, we see that it suffices to show that each $T_{m-1}^k F'(X) \to T_{m-1}^{k+1} F'(X)$ factors through a final object. It isn't hard to see that viewing $F$ as an $m$-variable functor, it being $(1, ..., 1)$ reduced implies the same for $T_{m-1}F$. This means that it suffices to show that for any $(1, ..., 1)$-reduced functor $G$ (which we denote $G'$ when we see it as a single variable functor) the natural map $\theta : G'(X) \to T_{m-1}G'(X)$ does so.
Let $X = (X_1, ..., X_m) \in \mathrm{Ob}(\mathcal{C})$ be some fixed object of $\mathcal{C}^n$. Recalling the definitions, we see that $\theta$ is equivalent to the natural map

$$G'(X) = G'(C_\emptyset(X)) \to \varprojlim_{\emptyset \neq S \subset [m-1]} G'(C_S(X)).$$

However in order to use the assumption that $G$ is reduced in each variable, it is natural to want to write the above by considering $G$ as a multivariable functor. This is done by the following functor $Y : \mathrm{N}(\mathcal{P}([m-1])^m) \to \mathcal{D}$ which on objects is defined by

$$(S_1, ..., S_m) \mapsto G(C_{S_1}(X_1), ..., C_{S_m(X_m)}).$$

Note we are mildly abusing notation and identifying $[m-1]$ with $\{1, ..., m\}$ in order for the subscripts of the sets $S_\bullet$ to match the subscripts of the objects $X_\bullet$. With this functor in hand, we can rewrite the map of interest as

$$Y(\emptyset, ..., \emptyset) \to \varprojlim_{\mathrm{N}(\mathcal{P}([m-1])^m)} Y \to \varprojlim_{\mathrm{N}(A)} Y|_{\mathrm{N}(A)},$$

where $A \subset \mathrm{N}(\mathcal{P}([m-1])^m)$ is the full subcategory consisting of tuples where every object is the same set. In order to see that this map factors through a final object, we will find some intermediary diagram $A \subset B \subset \mathrm{N}(\mathcal{P}([m-1])^m)$ with the property that $\varprojlim_{\mathrm{N}(B)} Y|_{\mathrm{N}(B)}$ is a final object. Indeed, for such a $B$, we have that $\theta$ factors as

$$Y(\emptyset, ..., \emptyset) \to \varprojlim_{\mathrm{N}(\mathcal{P}([m-1])^m)} Y \to \varprojlim_{\mathrm{N}(B)} Y|_{\mathrm{N}(B)} \simeq * \to \varprojlim_{\mathrm{N}(A)} Y|_{\mathrm{N}(A)}.$$

We let $B$ be the full subcategory spanned by those $m$-tuples $(S_1, ..., S_m)$ such that there exists an index $i$ with $i \in S_i$. The required property that $A \subset B$ is clear. Now to compute the desired limit we use the recurring trick of finding a different diagram with equivalent but easier to compute limit. For this, let $B_0 \subset B$ consist of those $m$-tuples such that for some index $i$ we have $S_i = \{i\}$. By using theorem A.0.10, we see that to show that this map is initial, it suffices to show that for each $(S_1, ..., S_m) \in B$, the nerve of the full subcategory $V$ whose objects are

$$\{(S'_1, ..., S'_m) \in B_0 | S'_i \subset S_i\}$$

is weakly contractible. We have an inclusion $V_0 \subset V$, where $V_0$ consists of those $m$-tuples such that $S'_i \subset \{i\}$. This inclusion admits a right adjoint as it clearly preserves all supremums (i.e. colimits), so that we may conclude by the adjoint functor theorem for posets. Thus $N(V_0) \simeq N(V)$. As $V_0$ has a final object $(\{1\}, \{2\}, ..., \{m\})$, we have that $N(V_0)$ is contractible. Thus the restriction map $\varprojlim_{\mathrm{N}(B)} Y|_{\mathrm{N}(B)} \to \varprojlim_{\mathrm{N}(B_0)} Y|_{\mathrm{N}(B_0)}$ is an equivalence and this limit is easy to compute.
Indeed, for each vertex $(S_1, ...S_m) \in B_0$, we have that at least one of the $S_i$ is a singleton, so that $C_{S_i}(X_i)$ is a final object. Which because $G$ is reduced in each variable by assumption implies that $Y(S_1, ..., S_m) = G(C_{S_1}(X_1), ..., C_{S_i}(X_i), ..., C_{S_m}(X_m))$ is final. A diagram of final objects is final, thus showing that $\varprojlim_{\mathrm{N}(B)} Y|_{\mathrm{N}(B)} \simeq *$. This proves the desired result. □

Note that the above two results can be combined to say that a functor $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ which is 1-homogeneous in each variable is $m$-homogeneous when viewed as a single variable functor.

The property of being reduced can be quite desirable, so it would be nice to have have a functorial way to obtain reduced functors from arbitrary functors. This is accomplished by the following result.

**Proposition 2.1.6.** *(Corollary 6.1.3.18. in [21]) Let $\mathcal{C}_1, ..., \mathcal{C}_m$ be categories which admit a final object and let $\mathcal{D}$ be a pointed category with finite limits. Then the inclusion*

$$\mathrm{Fun}_*(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$$

*admits a right adjoint denoted by* $\mathrm{Red} : \mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}_*(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$.

Using that the subcategory of reduced functor is full, it will suffice to find a functor $\mathrm{Red} : \mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D}) \to \mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$ such that $\mathrm{Red}(F)$ is always a reduced functor, and equipped with a natural map $\mathrm{Red}(F) \to F$ which induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, \mathrm{Red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F).$$

We explicitly construct Red in the following definition, and then the following proposition will show the desired properties so that this functor is a right adjoint to the inclusion.

**Definition 2.1.7.** (Construction 6.1.3.15. in [21]) Let $\mathcal{C}_1, ..., \mathcal{C}_m$ be categories which admit a final object $*_i \in \mathcal{C}_i$, let $\mathcal{D}$ be a pointed category with finite limits and let $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ be a functor. For each $i$, denote by $U_i : \mathcal{C}_i \to \mathcal{C}_i$ the constant functor equal to $*_i$. Choose a natural transformation $\alpha_i : \mathrm{Id}_{\mathcal{C}_i} \to U_i$. Let $T \subset [m]$, and define $F^T(X_1, ..., X_m) = F(X'_1, ..., X'_m)$ where $X'_i = X_i$ if $i \notin T$ and $X'_i = *_i$ if $i \in T$. Using the $\alpha_i$ we can assemble these functors into a $N(\mathcal{P}([m]))$-shaped diagram of functors.

Using this diagram it is clear that we have a natural transformation $\beta : F = F^{\emptyset} \to \varprojlim_{\emptyset \neq T \subset [m]} F^T$. Because $\mathcal{D}$ is pointed, the same holds true for the functor category $\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})$, and so we can take the fiber of $\beta$. This fiber is what we call the reduction of $F$ and denote by $\mathrm{Red}(F)$. This functor, as the fiber of a map $F \to \varprojlim_{\emptyset \neq T \subset [m]} F^T$ admits a natural map to $F$.

**Lemma 2.1.8.** *(Proposition 6.1.3.17. in [21]) Let $\mathcal{C}_1, ..., \mathcal{C}_m$ be categories which admit a final object $*_i \in \mathcal{C}_i$, let $\mathcal{D}$ be a pointed category with finite limits and let $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ be a functor. Then* $\mathrm{Red}(F)$ *is a reduced functor and the natural map* $\mathrm{Red}(F) \to F$ *induces an equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, \mathrm{Red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F).$$

*Proof.* We use the same notation as in the statement of the result and the definition of $\mathrm{Red}(F)$, in particular we use without further comment the notation $F^T$ for $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ a functor and $T \subset [m]$ a subset. At first we want to show that $\mathrm{Red}(F)$ is a reduced functor. To do this let $(X_1, ..., X_m) \in \mathrm{Ob}(\prod_{i=1}^m \mathcal{C}_i)$ be such that at least one of the $X_i$ is a final object, say $X_j$. If we show $\mathrm{Red}(F)(X_1, ..., X_m)$ is a final object, we are done. Because $\mathrm{Red}(F)$ is a fiber of some map, it suffices to show the map $\beta : F = F^{\emptyset} \to \varprojlim_{\emptyset \neq T \subset [m]} F^T$ in question is an equivalence, which is what we will prove.
To this end, notice at first that for all $T \subset [m]$, the natural map $F^T(X_1, ..., X_m) \to F^{T \cup \{j\}}(X_1, ..., X_m)$ is an equivalence. From this we see that the $m$-cube $T \mapsto F^T(X_1, ..., X_m)$ is a right Kan extension of the restriction of this diagram to the subcategory of $\mathrm{N}(\mathcal{P}([m]))$ containing $j$. So the natural map

$$\varprojlim_{\emptyset \neq T \subset [m]} F^T(X_1, ..., X_m) \to \varprojlim_{\{j\} \subset T \subset [m]} F^T(X_1, ..., X_m)$$

is an equivalence, further one may notice that the codomain of this equivalence can be chosen to be $F^{\{j\}}(X_1, ..., X_m)$ as $j$ is initial in the subdiagram of $\mathrm{N}(\mathcal{P}([m]))$ consisting of those subsets containing $j$. So we can compute $\mathrm{Red}(F)(X_1, ..., X_m)$ as the fiber of the map $F(X_1, ..., X_m) \to F^{\{j\}}(X_1, ..., X_m)$, but as $X_j$ is a final object, this map is an equivalence so that $\mathrm{Red}(F)(X_1, ..., X_m)$ is a 0 object, in particular final, proving that $\mathrm{Red}(F)$ is final.

We now prove that the natural map $\mathrm{Red}(F) \to F$ induces the desired equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, \mathrm{Red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F)$$

when $G$ is a reduced functor. Recall that because $\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, -)$ preserves limits, it in particular preserves fiber sequences. Namely, it sends the fiber sequence defining $\mathrm{Red}(F)$ to the fiber sequence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, \mathrm{Red}(F)) \to \mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F) \to \varprojlim_{\emptyset \neq T \subset [m]} \mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F^T).$$

As a limit of contractible spaces is contractible, this reduces the problem to showing that for each $T \neq \emptyset$ the space $\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F^T)$ is contractible. To do this, et $j \in T$ and consider the subcategory $\mathcal{E} \subset \prod_{i=1}^m \mathcal{C}_i$ of $m$-tuples where the $j$th coordinate is a final object. The interest of this subcategory is that, because $G$ is a reduced functor, it is a natural subcategory for which when we restrict, it is obvious that $\mathrm{Map}_{\mathcal{E}}(G|_{\mathcal{E}}, F^T|_{\mathcal{E}})$ is contractible. In order for this to be of any use, we would want the restriction $\mathrm{Map}_{\mathrm{Fun}(\prod_{i=1}^m \mathcal{C}_i, \mathcal{D})}(G, F^T) \to \mathrm{Map}_{\mathcal{E}}(G|_{\mathcal{E}}, F^T|_{\mathcal{E}})$ to be an equivalence. This will be the case if $F$ is a right Kan extension of its restriction to $\mathcal{E}$ by lemma A.0.5. But this is obvious by construction of $F^T$, thus concluding the proof. $\square$

This proves that the reduction functor we constructed is a right adjoint to the inclusion of reduced functors, thus proving proposition 2.1.6. This now allows us to define the cross effect of a functor.

**Definition 2.1.9.** (Construction 6.1.3.20. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed category with finite limits, and $F : \mathcal{C} \to \mathcal{D}$. We have a functor $q : \mathcal{C}^n \to \mathcal{C}$ which maps an $n$-tuple to the coproduct of these objects. The functor $\mathrm{cr}_n := \mathrm{Red}(F \circ q) : \mathcal{C}^n \to \mathcal{D}$ is defined, this is what we call the *nth-cross effect*.

For example, for $n = 2$, the cross effect of a functor $F : \mathcal{C} \to \mathcal{D}$ evaluated at $(c_1, c_2)$ is the fiber of the natural map $F(c_1 \sqcup c_2) \to F(c_1) \times_{F(\emptyset)} F(c_2)$.

Intuitively, it is clear in what sense the $n$th cross-effect of a functor is a symmetric functor, our goal for the remainder of this section is to make this precise. Our first step towards this is to define an analogue of $\mathcal{C}^n$ whose $n$-tuples aren't ordered.

**Definition 2.1.10.** (Notation 6.1.4.1. in [21]) Recall that for a group $G$, we have a contractible simplicial set $EG$ with a free action of $G$, such that $EG/G \simeq BG$ is a classifying space. Then for a simplicial set $K$, the simplicial set

$$(K^n \times E\Sigma_n)/\Sigma_n$$

is an explicit model for the (homotopy) limit $K^n/\Sigma_n$. In the case where $K$ is an $\infty$-category, this quotient does indeed encode unordered $n$-tuples of $K$. We denote this construction by $K^{(n)}$.
This allows us to define a symmetric $n$-ary functor as a functor $\mathcal{C}^{(n)} \to \mathcal{D}$, these obviously assemble into a category $\mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. A symmetric $n$-ary functor always has an underlying functor $\mathcal{C}^n \to \mathcal{D}$, and so we define a symmetric $n$-ary functor to be reduced if it is underlying functor is reduced (in each variable). The reduced symmetric $n$-ary functors assemble into a categoy $\mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D})$ which is a full subcategory of $\mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$.

Now to find ourselves in a situation where we can say that the cross effect is naturally symmetric, we would want a symmetric equivalent of proposition 2.1.6. This is accomplished by the following result.

**Proposition 2.1.11.** *(Proposition 6.1.4.3. and remark 6.1.4.4. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object and let $\mathcal{D}$ be a pointed category with finite limits. Then, the inclusion* $\mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$ *admits a right adjoint $\theta$ which fits into the following diagram*

$$\begin{tikzcd} \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \arrow[r, "\theta"] \arrow[d] & \mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathrm{Fun}(\mathcal{C}^n, \mathcal{D}) \arrow[r, "\mathrm{Red}"] & \mathrm{Fun}_*(\mathcal{C}^n, \mathcal{D}) \end{tikzcd} .$$

*The vertical maps are the passage to the induced functor.*

*Proof.* The inclusion of categories $\iota : \mathrm{Fun}_*(\mathcal{C}^n, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}^n, \mathcal{D})$ is in fact a $\Sigma_n$-equivariant map, and quotienting by this action yields the inclusion we are interested in $\iota_{\mathrm{Sym}} : \mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. From this a natural guess for the right adjoint of $\iota_{\mathrm{Sym}}$ is the map induced by Red after quotienting by $\Sigma_n$. This makes sense as Red is $\Sigma_n$-equivariant. Call this functor $\theta$, we wish to show that $\theta$ is a right adjoint to $\iota_{\mathrm{Sym}}$. This follows from the fact that a natural transformation of two functors $F, G : \mathcal{C}^{(n)} \to \mathcal{D}$ contains the same data as a natural transformation of the underlying functors $\mathcal{C}^n \to \mathcal{D}$.
The fact that $\theta$ and Red fit into a commutative diagram as in the statement of the result can be seen directly by observing what happens to a specific $F \in \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D})$. □

Although they are two different functors, we will abuse notation and write Red both for the functor $\mathrm{Fun}(\mathcal{C}^n, \mathcal{D}) \to \mathrm{Fun}_*(\mathcal{C}^n, \mathcal{D})$ and for the functor $\mathrm{SymFun}(\mathcal{C}^n, \mathcal{D}) \to \mathrm{SymFun}_*(\mathcal{C}^n, \mathcal{D})$. If we need to be specific we may denote the latter by $\mathrm{Red}_{\mathrm{Sym}}$.

Now because the map $F \circ q(X_1, ..., X_n) \to F(X_1 \sqcup ... \sqcup X_n)$ is symmetric, it makes sense that we should take its symmetric reduction, thus defining the symmetric cross effect.

**Definition 2.1.12.** (Construction 6.1.4.5. in [21]) Given a functor $F : \mathcal{C} \to \mathcal{D}$ from a category with finite colimits and a final object to a pointed category with finite limits, we denote by $\mathrm{cr}_{(n)}$ the symmetric reduction of $F \circ q : \mathcal{C}^{(n)} \to \mathcal{D}$.

We allowed ourselves the abuse of notation of writing $q$ both for the functor $\mathcal{C}^n \to \mathcal{C}$ and for the functor $\mathcal{C}^{(n)} \to \mathcal{C}$. This shouldn't cause any confusion. Obviously the symmetric cross effect is a symmetric functor, so has an underlying functor $\mathcal{C}^n \to \mathcal{D}$. It isn't hard to see that this functor is the "ordinary" cross effect and how they fit into the following diagram

$$\begin{tikzcd} & \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \arrow[d] \\ \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[ur, "\mathrm{cr}_{(n)}"] \arrow[r, "\mathrm{cr}_n"] & \mathrm{Fun}(\mathcal{C}^n, \mathcal{D}) \end{tikzcd}.$$

As we wish to study the cross effect, it is natural to study the excisiveness, when the input functor itself is excisive, this is accomplished by the following result. It is clear how to interpret this result for the symmetric cross effect, by passing to the underlying functor.

**Proposition 2.1.13.** *(Proposition 6.1.3.22. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category and let $F : \mathcal{C} \to \mathcal{D}$ be an $n$-excisive functor. For each $m \leq n+1$, we have that $\mathrm{cr}_m(F) : \mathcal{C}^m \to \mathcal{D}$ is $(n-m+1)$-excisive in each variable.*

*Proof.* As the condition is vacuous for $m = 0$, we are encouraged to pursue a proof by induction. Assume the result holds for $m-1$, let's show it for $m$. Because the cross effect is invariant under permutation of its input, we may fix $X_2, ..., X_m$ and simply show that $X_1 \mapsto \mathrm{cr}_m(F)(X_1, ..., X_m)$ is $(n-m+1)$-excisive, to show that $\mathrm{cr}_m$ is $(n-m+1)$-excisive in each variable. For this, let $*$ be a final object of $\mathcal{C}$, let $G(X) = F(X \sqcup X_m)$, $G''(X) = (X \sqcup *)$ and $G'(X)$ the fiber of the natural map $G(X) \to G''(X)$ induced by the map $X_m \to *$. Because limits commute, consulting the definition, it isn't hard to see that

$$\mathrm{cr}_m(F)(X_1, ..., X_m) \simeq \mathrm{cr}_{m-1} G'(X_1, ..., X_{m-1}).$$

So to show the $(n+m-1)$-excisiveness of the functor on the left, it suffices to prove this for the functor on the right. And for this, by the induction hypothesis, it suffices to show that $G'$ is $(n-1)$-excisive. Let $Y$ be a strongly coCartesian $(n-1)$-cube in $\mathcal{C}$, we need to show that $G'(Y)$ is Cartesian. We define an $n$-cube $Y'$ by mapping a subset $T \subset [n]$ to $Y(T) \sqcup X_m$ if $n \notin T$ and to $Y(T \backslash \{n\}) \sqcup *$ if $n \in Y$. Using lemma 1.1.4 we see that this cube is still strongly coCartesian, so $F(Y')$ is a Cartesian cube. This means that the following square is a pullback

$$\begin{tikzcd} F(Y(\emptyset) \sqcup X_m) \arrow[r] \arrow[d] & \varprojlim_{\emptyset \neq S \subset [n-1]} F(Y(\emptyset) \sqcup X_m) \arrow[d] \\ F(Y(\emptyset) \sqcup *) \arrow[r] & \varprojlim_{\emptyset \neq S \subset [n-1]} F(Y(\emptyset) \sqcup *) \end{tikzcd}.$$

We obtain the desired result by taking fibers of the vertical maps, which are equivalent as this square is a pullback. $\square$

## 2.2 Classifying homogeneous functors

In keeping with the analogy that we are trying to develop a parallel to the theory of derivatives and Taylor series of ordinary calculus, one can notice that we have done things "upside down". We have defined a satisfactory notion of "degree $n$ polynomial approximation", and from this trying to obtain a satisfactory definition of $n$th derivative. It isn't too hard to "subtract" to approximate a degree $(n-1)$-approximation from a degree $n$-approximation, by taking the fiber of the natural map $P_nF \to P_{n-1}F$. This gives us an analogue to the association $f \mapsto \frac{f^{(n)}(0)}{n!}x^n$ in classical calculus, and so all that is left to do is to multiply by $n!$ (and then evaluate at x$x = 1$). This is surprisingly hard, because there is no obvious candidate for $n! \cdot (-)$ in this abstract framework.
However, in classical calculus, there is another construction which can get rid of the $n!$, and this is what we emulate in this section. Given a continuous function $f : \mathbb{R} \to \mathbb{R}$, associate to it its "$n$th cross effect", which is given by

$$\mathrm{cr}_n(f)(x_1, ..., x_n) = \Sigma_{i=0}^n (-1)^{n-i} \Sigma_{I \subset \{1,...,n\}, |I|=i} f(\Sigma_{k \in I} x_k).$$

Although quite convoluted, if we observe what this map does to monomials, and polynomials of degree $n$, one might end up considerably less surprised by theorem 2.2.1 and lemma 2.2.4.
Indeed, one can observe by induction that $\mathrm{cr}_n(ax^n) = a(n!x_1x_2 \cdots x_n)$, which is a convoluted way to multiply by $n!$, though in our situation turns out to be more natural, and is also an association of a multilinear polynomial to a homogeneous degree $n$ polynomial, which is exactly what theorem 2.2.1 will achieve. One can also observe, that a polynomial $p$ of degree $n$ such that $\mathrm{cr}_n(p) = 0$ is at most of degree $(n-1)$, which is a classical analogue of lemma 2.2.4.

For us, these two results are logically dependent as the latter will serve to prove former, whose proof is the main goal of this subsection. The statement of this result in our abstract setting is as follows.

**Theorem 2.2.1.** *(6.1.4.7. in [21]) Let $\mathcal{C}$ be a pointed category with finite colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category. Then we have a fully faithfull embedding*

$$\mathrm{cr}_{(n)} : \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \to \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}).$$

*The essential image of $\mathrm{cr}_{(n)}$ is the full subcategory $\mathrm{SymFun}^n_{lin}(\mathcal{C}, \mathcal{D})$ of those functors $E : \mathcal{C}^{(n)} \to \mathcal{D}$ whose underlying functor $E : \mathcal{C}^n \to \mathcal{D}$ is multilinear.*

The proof of this result will rely on a reduction to the stable case, courtesy of corollary 1.3.3, on the multivariable theory developed in subsection 2.1 and on a couple of extra lemmas we prove in this section.

We first prove a series of lemmas with the end of goal of proving that $\mathrm{cr}_{(n)}$ is conservative, i.e. preserves and reflects equivalences. Recall that our proof will rely on reducing to the stable case, which is why (some of) our lemmas are concerned with functors with stable codomain.

**Lemma 2.2.2.** *(6.1.4.8. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a category with finite limits. And let $F : \mathcal{C} \to \mathcal{D}$ be an $n$-excisive functor, with $n \geq 1$. We have an equivalence between*

*(i) $F$ is $(n-1)$-excisive and*

*(ii) let $X$ be a strongly coCartesian $(n-1)$-cube such that $X(\emptyset)$ is a final object of $\mathcal{C}$, then $F(X)$ is a Cartesian $(n-1)$ cube of $\mathcal{D}$.*

*Proof.* The implication $(i) \Rightarrow (ii)$ is immediate, so assume $F$ is an $n$-excisive functor which carries strongly coCartesian $(n-1)$-cubes $Y$ with $Y(\emptyset)$ final to Cartesian cubes. We wish to show this functor carries any strongly coCartesian $(n-1)$-cube to a Cartesian cube. So let $X$ be a strongly coCartesian $(n-1)$-cube. We can consider a strongly coCartesian $n$-cube $X'$ with the property that

$X'|_{\mathrm{N}(\mathcal{P}([n-1]))} = X$ and $X'(\{n\}) = *$ is a final object of $\mathcal{C}$. Now by assumption that $F$ is $n$-excisive, we have that $FX'$ is a Cartesian cube. Thus we have a pullback square:

$$\begin{array}{ccc} F(X(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq S \subset [n-1]} F(X(S)) \\ \downarrow & & \downarrow \\ F(X(\{n\}) & \longrightarrow & \varprojlim_{\emptyset \neq S \subset [n-1]} F(X(S \cup \{n\})) \end{array} .$$

Now the bottom map is the natural map $F(\tilde{X}(\emptyset)) \to \varprojlim_{\emptyset \neq S \subset [n-1]} F(\tilde{X}(S))$, where $\tilde{X} : S \mapsto X'(S \cup \{n\})$ is strongly coCartesian $(n-1)$-cube with $\tilde{X}(\emptyset)$ final by construction. By assumption, $F$ sends $\tilde{X}$ to a Cartesian cube, thus this map is an equivalence. And as the pullback of an equivalence is an equivalence, we are done. □

**Lemma 2.2.3.** *(Lemma 6.1.4.9. in [21]) let $\mathcal{C}$ be a category with finite colimits and a final object $*$, let $\mathcal{D}$ be a stable category and let $F : \mathcal{C} \to \mathcal{D}$ be an $n$-excisive functor, with $n \geq 1$. Recall that $q : \mathcal{C}^n \to \mathcal{C}$ is the coproduct functor. We have an equivalence between*

*(i) $F$ is $(n-1)$-excisive and*

*(ii) for every finite sequence of morphisms $\{* \to C_i\}_{i=1}^n$, which we consider as maps $\alpha_i : \Delta^1 \to \mathcal{C}$, the natural strongly coCartesian $(n-1)$-cube $X$ (seen as a map $\mathrm{N}(\mathcal{P}(\{1, ..., n\})) \to \mathcal{C}$ for convenience of notation) with the property that $X(\emptyset \to \{i\}) = * \to C_i$ is sent to a C artesian cube by $F$.*

*Proof.* Once again the implication $(i) \Rightarrow (ii)$ is immediate, so assume that $(ii)$ holds. To show $(n-1)$-excisiveness from $n$-excisiveness, we now have the above lemma. Let $Y$ be a strongly coCartesian $(n-1)$-cube, with the property that $Y(\emptyset)$ is final, if we show that $F(Y)$ is Cartesian, we are done. Now let $C_i = Y(\{i\})$, we have natural maps $* \to C_i$ where $*$ is a final object. These maps can be seen as maps $\alpha_i : \Delta^1 \to \mathcal{C}$, which assemble, as in the statement of the lemma, into a strongly coCartesian cube $X$. There is clearly a natural map $\beta : X \to Y$, and all of this can be assembled into a strongly coCartesian cube $n$-cube $Z$. Now because $F$ is $n$-excisive, it sends $Z$ to a Cartesian cube, which in turn implies we have a pullback

$$\begin{array}{ccc} F(X(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq S \subset [n-1]} F(X(S)) \\ \downarrow & & \downarrow \\ F((Y(\emptyset)) & \longrightarrow & \varprojlim_{\emptyset \neq S \subset [n-1]} F(Y(S)) \end{array} .$$

Now the top map is an equivalence by assumption, this square is also a pushout as $\mathcal{D}$ is stable and the pushout of an equivalence is an equivalence, thus proving the claim. □

**Lemma 2.2.4.** *(Lemma 6.1.4.10. in [21]) let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a stable category and let $F : \mathcal{C} \to \mathcal{D}$ be an $n$-excisive functor, with $n \geq 1$. We have an equivalence between*

*(i) $F$ is $(n-1)$-excisive and*

*(ii) The $n$-fold cross effect $\mathrm{cr}_n(F)$ maps every object of $\mathcal{C}^n$ to the 0-object of $\mathcal{D}$.*

*Proof.* If $F$ is $(n-1)$-excisive, by lemma 2.1.13 we may conclude that $\mathrm{cr}_n(F)$ is a $(0, ..., 0)$ excisive functor, or in other words it is constant. As it is also reduced, it must be constant equal to a final object. Now assume that $F$ is $n$-excisive and that $\mathrm{cr}_n(F)$ is constant equal to a final object.
We will use the previous lemma. Let $\{\alpha_i : * \to C_i\}_{i=1}^n$ be a collection of maps, where $*$ is a final object of $\mathcal{C}$. Let $X$ be the natural strongly coCartesian $n-1$, viewed as a map $\mathrm{N}(\mathcal{P}(\{1, ..., n\})) \to \mathcal{C}$ for convenience of notation, with the property that $X(\emptyset \to \{i\}) = * \to C_i$. If $F(X)$ is Cartesian, we

are done.
For each $1 \leq i \leq n$, extend $\alpha_i$ to the following two simplex $\sigma_i$

$$\begin{tikzcd} & C_i \arrow[dr, "!"] & \\ * \arrow[ur, "\alpha_i"] \arrow[rr, "Id"'] & & * \end{tikzcd} .$$

Denoting the set $\{1, ..., n\}$ by $S$, we have a natural map $Y : (\Delta^2)^S \xrightarrow{\prod_{i=1}^n \sigma_i} \mathcal{C}^S \xrightarrow{q} \mathcal{C}$ where $q$ is the coproduct map. We use $Y$ to create cubes which interpolate between a cube which obviously mapped to a Cartesian cube by $Y$ and the cube $X$ of interest to us. We do this in order to set up a proof by induction. This interpolation is defined as follow, let $Y_i : \mathrm{N}(\mathcal{P}(S)) \to \mathcal{C}$ be defined by $Y_i(T) = Y(a_1, ..., a_n)$ where $a_j = 0 \in \Delta^2$ if $j \geq i$ and $j \notin T$, $a_j = 2 \in \Delta^2$ if $j < i$ and $j \in T$ and $a_j = 1 \in \Delta^2$ in the remaining cases.
One can observe that $Y_n$ is equivalent to $X$, so we may now proceed with the promised induction that $F(Y_i)$ is a Cartesian cube for each $i$. For $i = 0$, we need to show that the map

$$F(Y_0(\emptyset)) \simeq \varprojlim_{\emptyset \neq T \subset S} (F(Y_0))$$

is an equivalence. It isn't hard to notice that the fiber of this map is $\mathrm{cr}_n(F)$, which vanishes by assumption, thus implying that the above map is indeed an equivalence.
Now for the induction step, we may suppose the result holds for $i - 1 \geq 0$, and we want to show it for $i$. Let $S' = S \backslash \{i\}$, we have a commutative diagram

$$\begin{tikzcd} F(Y_i(\emptyset)) \arrow[r] \arrow[d] & F(Y_{i-1}(\emptyset)) \arrow[r] \arrow[d] & F(Y_{i-1}(\{i\})) \arrow[d] \\ \varprojlim_{\emptyset \neq T \subset S'} F(Y_i(T)) \arrow[r] & \varprojlim_{\emptyset \neq T \subset S'} F(Y_{i-1}(T)) \arrow[r] & \varprojlim_{\emptyset \neq T \subset S'} F(Y_{i-1}(T \cup \{i\})) \end{tikzcd} .$$

The right most square is a pullback by induction assumption. Also by construction the horizontal compositions are equivalences, so that the big outer rectangle is a pullback as well. This is known to imply that the left square is a pullback. We obtain the desired result after noticing that $Y_{i-1}(T) = Y_i(T \cup \{i\})$, so that the statement the left square is a pullback implies that $F(Y_i(\emptyset)) \to \varprojlim_{\emptyset \neq T \subset S} F(Y_i(T))$ is an equivalence. $\square$

**Lemma 2.2.5.** *(Lemma 6.1.4.11. in [21]) Let $\mathcal{C}$ be a pointed category with finite colimits, let $\mathcal{D}$ be a stable category which admits finite colimits. Let $\alpha : F \to G$ be a natural transformation of functors $F, G : \mathcal{C} \to \mathcal{D}$. Then $\alpha$ is an equivalence if and only if $\mathrm{cr}_n(\alpha)$ is an equivalence.*

*Proof.* Assume that $\mathrm{cr}_n(\alpha)$ is an equivalence, let $H$ be the fiber of $\alpha$. The fiber of a map of $n$-excisive functors is $n$-excisive, because $P_n$ is left exact and the fiber of a map of $n$-reduced functors is $n$-reduced for the same reason. So $H$ is $n$-homogeneous. Our goal is to use the previous lemma to show that $H$ is $(n-1)$-excisive, which because $H$ is $n$-reduced will imply that $H$ is the 0 of the stable category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, which will imply that $\alpha$ is an equivalence, as desired.
So we want to compute $\mathrm{cr}_n(H)$. For this observe that $H(X_1 \sqcup ... \sqcup X_n) = \mathrm{Fib}(F \xrightarrow{\alpha} G)(X_1 \sqcup ... \sqcup X_n) \simeq \mathrm{Fib}(F(X_1 \sqcup ... \sqcup X_n) \xrightarrow{\alpha} G(X_1 \sqcup ... \sqcup X_n))$. Combining this with the fact that Red is a right adjoint, so preserves fibers, yields that $\mathrm{cr}_n H \simeq \mathrm{Fib}(\mathrm{cr}_n(\alpha))$. Because $\mathrm{cr}_n(\alpha)$ is an equivalence, this shows that $\mathrm{cr}_n H$ must be constant equal to the 0-object of the stable category $\mathcal{D}$ or in other words is the 0-object of the stable category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. This is what we wanted to show. $\square$

The next two lemmas will in time serve to show that the functor $\mathrm{cr}_{(n)}$ is an adjoint, which will be an important part in the proof of theorem 2.2.1.

**Lemma 2.2.6.** *(Lemma 6.1.4.12. in [21]) Let $\mathcal{C}$ be a category with finite colimits, let $\mathcal{D}$ be a stable category and let $F : \mathcal{C} \to \mathcal{D}$ be a 1-excisive functor. Then $F$ carries strongly coCartesian cubes to strongly coCartesian cubes.*

*Proof.* This is an immediate corollary to lemma 1.1.4. Indeed $F$ carries pushouts to pullbacks, which are themselves pushouts as $\mathcal{D}$ is stable, and from this, one may verify immediately using one of the alternative characterizations of strongly coCartesian that $F$ preserves general strongly coCartesian cubes. $\square$

**Lemma 2.2.7.** *(Lemma 6.1.4.13. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object, let $\mathcal{D}$ be a stable category and let $F : \mathcal{C}^n \to \mathcal{D}$ be a $(1, ..., 1)$-excisive functor. For every $\sigma \in \Sigma_n$ let $F^\sigma$ be the precomposition by the natural automorphism $\mathcal{C}^n \to \mathcal{C}^n$ which permutes the factors via $\sigma$. Let $\delta : \mathcal{C} \to \mathcal{C}^n$ be the diagonal map, $f = F \circ \delta$. Then there is a canonical equivalence*

$$\mathrm{cr}_n(f) \simeq \bigoplus_{\sigma\in\Sigma_n} \mathrm{Red}(F^\sigma).$$

*In particular if $F$ is $(1, ..., 1)$-homogeneous, then $\mathrm{cr}_n(f) \simeq \bigoplus_{\sigma\in\Sigma_n} F^\sigma$.*

*Proof.* It is a recurring idea to prove results by "fattening" points by sets. In this proof, letting $S = \{1, ..., n\}$, we do this by introducing for all $\vec{T} = (T_1, ..., T_n) \in \mathrm{N}(\mathcal{P}(S))$ the functor $F_{\vec{T}}$, which is precomposition of $F$ by the functor $\mathcal{C}^n \to \mathcal{C}^n$, which is defined as

$$(X_1, ..., X_n) \mapsto (\bigsqcup_{i\in T_1} X_i, \bigsqcup_{i\in T_2} X_i, ..., \bigsqcup_{i\in T_n} X_n).$$

The sense in which this is a "fattening" of the construction $F^\sigma$, which associates to a map $S \to S$ a modified version of $F$, is that taking $\vec{T_\sigma} = \{\sigma(1), ..., \sigma(n)\}$ yields $F^\sigma$; but in addition, with the construction just decribed we get a modified version of $F$ for any map $S \to \mathcal{P}(S)$. Further notice that for $\vec{T} = (S, ..., S)$, we have $\mathrm{Red}(F_{\vec{T}}) = \mathrm{cr}_n(f)$.
By the previous lemma, the assumption that $F$ is 1-excisive in each variable implies that, for any $(X_1, ..., X_n) \in \mathcal{C}^n$, the construction $\vec{T} \mapsto F_{\vec{T}}(X_1, ..., X_n)$ is strongly coCartesian separately in each variable. This implies that the following map is an equivalence

$$\varinjlim_{\vec{T}\in\mathrm{N}(\mathcal{P}_{\leq 1}(S)^n)} F_{\vec{T}} \to F_{(S,...,S)}.$$

Because $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is stable, the colimit of the map $Z : \mathrm{N}(\mathcal{P}_{\leq 1}(S)^n) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ defined by $Z(\vec{T}) = \mathrm{Red}(F_{\vec{T}})$ is $\mathrm{cr}_n(f)$. This is because Red is a right adjoint, thus left exact, which in a stable category is equivalent to right exact (this proposition 1.1.4.1. in [21]).
We will compute this colimit by changing the diagram category, let $P \subset \mathcal{P}_{\leq 1}(S)^n$ be the full subcategory whose objects are those $n$-tuples whose union is $S$. If $\vec{T} \in \mathrm{Ob}(\mathcal{P}_{\leq 1}(S)\backslash S)$, then $F_{\vec{T}}$ is independent of one of its arguments, so that $\mathrm{Red}(F_{\vec{T}})$ is constant equal to a 0 object of $\mathcal{D}$. Thus $Z$ is a left Kan extension of its restriction to $\mathrm{N}(P)$, which means we can compute the colimit $\varinjlim_{\vec{T}\in\mathrm{N}(\mathcal{P}_{\leq 1}(S)^n)} \mathrm{Red}(F_{\vec{T}})$ after restricting to $\mathrm{N}(P)$. We can see this claim by pondering the following composing of adjunction (where the top map is the left adjoint) and noticing that $\iota^* \circ \Delta = \Delta$:

$$\begin{tikzcd} \mathrm{Fun}(P, \mathcal{F}) \arrow[r, bend left, "\iota_!"] & \mathrm{Fun}(\mathrm{N}(\mathcal{P}_{\leq 1}(S)^n), \mathcal{F}) \arrow[l, bend left, "\iota^*"] \arrow[r, bend left, "\varinjlim"] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[l, bend left, "\Delta"] \end{tikzcd} .$$

We denote by $\iota_!$ the left adjoint to precomposing by $\iota$, by proposition 4.3.2.17. in [20], this corresponds to left Kan extensions.
Now the result follows by noticing that elements of $\vec{T} \in \mathrm{N}(P)$ correspond to permutations of $S$ in such a way that $F_{\vec{T}} \simeq F^\sigma$ and that $P$ is a discrete poset, so that we have

$$\mathrm{cr}_n(f) \simeq \varinjlim_{\vec{T}\in\mathrm{N}(\mathcal{P}_{\leq 1}(S)^n)} \mathrm{Red}(F_{\vec{T}}) \simeq \varinjlim_{\vec{T}\in\mathrm{N}(P)} \mathrm{Red}(F_{\vec{T}}) \simeq \bigoplus_{\sigma\in\Sigma_n} F^\sigma.$$

$\square$

With all of these lemmas in hand, we are finally ready to prove theorem 2.2.1.

*Proof.* A priori, it isn't clear the the codomain is what we claim it to be. In particular, for $F$ an $n$-homogeneous functor, we want to be sure that $\mathrm{cr}_{(n)} F$ is symmetric and multilinear. It is symmetric by construction and multilinear by combining proposition 2.1.13 for $(1, ..., 1)$-excisiveness and 2.1.6 for reducedness.
Recall we have a functor $\Omega^\infty : \mathrm{Sp}(\mathcal{D}) \to \mathcal{D}$ (see notation 1.4.2.20. in [21]), this induces a commutative diagram

$$\begin{tikzcd} \mathrm{Homog}^n(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \arrow[r, "\mathrm{cr}_{(n)}"] \arrow[d] & \mathrm{SymFun}^n_{lin}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \arrow[d, "\psi"] \\ \mathrm{Homog}^n(\mathcal{C}, \mathcal{D}) \arrow[r, "\mathrm{cr}_{(n)}"] & \mathrm{SymFun}^n_{lin}(\mathcal{C}, \mathcal{D}) \end{tikzcd} .$$

We will use this diagram to reduce to the stable case. First, by corollary 1.3.3, the left vertical map is an equivalence. The right vertical map is also an equivalence due to corollary 1.3.3. To see this, first notice that by fixing all the coordinates but one, we obtain that $\mathrm{Exc}_*(\mathcal{C}^n, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}^n, \mathcal{D})$ is an equivalence, and from this we obtain the desired result by quotienting by the natural $\Sigma_n$ action. So to show the desired result that the bottom map is an equivalence, it suffices to show that the top map is an equivalence, thus we may henceforth assume that $\mathcal{D}$ is stable.
In particular, this means we may assume that $\mathcal{D}$ admits finite colimits (proposition 1.1.3.4. in [21]), which as $\mathcal{D}$ is assumed to contain sequential colimits implies it contains countable filtered colimits, which in turn implies $\mathcal{C}$ contains all countable limits (see 4.2.3.11. in [20]).
We will show that $\mathrm{cr}_{(n)}$ is an equivalence by showing that it is an adjoint equivalence, to do this we first construct a left adjoint to precomposition by $q$, i.e. a left Kan extension along $q$. Recall that $q : \mathcal{C}^{(n)} \to \mathcal{C}$ is the functor which maps an $n$-tuple to the coproduct. This can be rephrased as asking whether the induced maps $\mathcal{C}^{(n)} \times_{\mathcal{C}} \mathcal{C}_{/C} \to \mathcal{D}$ admit colimits for all $C \in \mathrm{Ob}(\mathcal{C})$. To show this, notice that the inclusion $\{C\}^{(n)} \to \mathcal{C}^{(n)} \times_{\mathcal{C}} \mathcal{C}_{/C}$ as the $n$-fold coproduct of the identity map exhibits $\{C\}^{(n)}$ as a final object in $\mathcal{C}^{(n)} \times_{\mathcal{C}} \mathcal{C}_{/C}$, in particular the inclusion is initial (i.e. by 7.2.3.7. in [22]). So it suffices for $\mathcal{D}$ to admit all $\{C\}^{(n)}$ shaped colimits. Inspecting the definition, this simplicial set is equivalent to $B\Sigma_n$ which contains countably many simplicies, and because $\mathcal{D}$ admits countable colimits, it admits $\{C\}^{(n)}$ shaped colimits. This proves that any functor $\mathcal{C}^{(n)} \to \mathcal{D}$ admits a left Kan extension along $q$. By proposition 4.3.2.17. in [20], the left Kan extension along $q$, denoted by $q_!$, is indeed the left adjoint to precomposition by $q$. We can fit this into the following diagram

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \arrow[rr, bend left, "-\circ q"] \arrow[rr, phantom, "\top"] && \mathrm{SymFun}^n(\mathcal{C}, \mathcal{D}) \arrow[ll, bend left, "q_!"] \arrow[rr, bend left, "\mathrm{Red}"] \arrow[rr, phantom, "\top"] && \mathrm{SymFun}^n_*(\mathcal{C}, \mathcal{D}) \arrow[ll, bend left, "\iota"] \end{tikzcd} .$$

As a composition of right adjoints is the right adjoint to the composition of the left adjoint, this shows that $\phi : q_! \circ \iota$ is a left adjoint to $\mathrm{cr}_{(n)} = \mathrm{Red}(- \circ q)$. We know that if we restrict the cross effect to $n$-homogeneous functors, we can corestrict to multilinear symmetric functors. In order to turn this adjunction into the desired equivalence, we also need to show that restricting $\phi$ to the multilinear symmetric functors, we may corestrict it to the $n$-homogeneous functors.
For this we need a slightly more explicit description of $\phi$, by studying the definition of a left Kan extension (see A.0.3) and using the initial map $B\Sigma_n \simeq \{C\}^{(n)} \to \mathcal{C}^{(n)} \times_{\mathcal{C}} \mathcal{C}_{/C}$, we see that pointwise $q_!(F)(C) = \varinjlim_{B\Sigma_n} F(C, ..., C) = F(C, ..., C)_{h\Sigma_n}$, which if we denote $\Delta : \mathcal{C} \to \mathcal{C}^n$ the diagonal map, can be stated as $q_! F = (F \circ \Delta)_{h\Sigma_n}$. As $\phi$ is just precomposition of $q_!$ by an inclusion the same computation holds for $\phi$. The functor $F \circ \Delta$ is $n$-reduced by lemma 2.1.5 and the fact that $F$ is multilinear and $n$-excisive by lemma 2.1.4 and again the fact that $F$ is multilinear, thus $F \circ \Delta$ is $n$-homogeneous. To see that $q_! F$ is itself $n$-homogeneous, it will suffice to show that when the codomain is stable, $n$-homogeneous is preserved by countable colimits. The fact that $n$-reducedness is preserved by colimits is immediate by the fact that colimits of functors can be computed pointwise and the fact that a

colimit of a diagram with 0 at every vertex is clearly 0. For $n$-excisiveness, this follows from the fact that $P_n$ is left exact by lemma 1.2.7, which in a stable category is equivalent to right exact (see 1.1.4.1. in [21]) and the fact that $P_n$ preserves sequential colimits. Indeed, this allows us to conclude that countable colimits are preserved by $P_n$ (by 4.2.3.12. in [20]), which directly implies that $n$-excisiveness is preserved by countable colimits. In particular, $n$-homogeneity is preserved by colimits over a $B\Sigma_n$ shaped diagram. All of this means that we can restrict the adjunction $\phi \vdash \mathrm{cr}_{(n)}$ in order to obtain an adjunction between $n$-homogeneous functors and symmetric in $n$-variable multilinear functors.
We will now show that this adjunction is in fact an adjoint equivalence, which will prove the desired claim. We already showed in lemma 2.2.5 that $\mathrm{cr}_{(n)}$ is a conservative functor, it isn't hard to see this reduces showing the equivalence to showing that the unit $\eta : \mathrm{Id} \to \mathrm{cr}_{(n)} \circ \phi$ is an equivalence. We can show this pointwise, i.e. for a specific functor show that we have an equivalence $F \to \mathrm{cr}_{(n)} \phi(F)$, we can show this for the underlying functors $\mathcal{C}^n \to \mathcal{D}$, i.e we want to show $F \to \mathrm{cr}_n \phi(F)$. We will allow ourselves the mild abuse of notation of writing $F$ for the functors out of $\mathcal{C}^{(n)}$ and the underlying functors out of $\mathcal{C}^n$. Now recalling that the cross effect is defined via a limit, and that in a stable category finite limits commute with colimits, because in presence of a terminal object all finite limits can be constructed from pullbacks, which obviously commute with colimits in the stable case, we see that $\mathrm{cr}_n \phi(F) \simeq \mathrm{cr}_n(F \circ \Delta)_{h\Sigma_n}$. By lemma 2.2.7, this is equivalent to $(\bigoplus_{\sigma\in\Sigma_n} \mathrm{Red}(F^\sigma))_{h\Sigma_n}$, but as $F$ is reduced, and the action of $\Sigma_n$ is by permuting the factors, it isn't hard to see that this is equivalent to $F$.
This is almost what we wanted to show, one still needs to observe that the unit $\eta$ realizes this isomorphism. This follows from comparing definitions, and so we don't detail this here. □

## 2.3 Equivalence of certain categories of multilinear functors

In this section we complete the final step in order to formulate and understand the definition of the $n$th derivative of a functor. A key step in this procedure is that we define an object corresponding to the first derivative prior to defining the $n$th derivative in full generality. Sadly, for us, the connection won't be as strong as in classical calculus, with the $n$th derivative simply an iterated application of the first, however this idea is pursued in [7].
Another key idea which we see the first hints of in this section, but will not at all pursue further, is that when defining $P_nF$ or the derivatives of $F$, leaving the domain and codomain unchanged isn't the most natural choice. Indeed, for example for $P_1F$, as this functor forcibly identifies pushouts with pullbacks, one might as well force this on the level of the domain and codomain. This idea is pursued in full in [15].
Before getting into the mathematics of this section, we inform the reader that the language of stable categories will be especially used in this section, and so one should be prepared to skimming through chapter 1 of [21].

The light at the end of this particular tunnel is the following result, which will be the final ingredient in defining the $n$th derivative of a functor.

**Theorem 2.3.1.** *(Proposition 6.2.3.21. and Corollary 6.2.3.22. in [21]) Let $\{\mathcal{C}_i\}_{i\in I}$ be a finite collection of pointed differentiable category and let $\mathcal{D}$ be a differentiable category. Then the construction $f \mapsto \Omega^\infty_{\mathcal{D}} \circ f \circ \prod_{i\in I} \Sigma^\infty_{\mathcal{C}_i}$ defines an equivalence $\phi : \mathrm{Exc}_\star(\prod_{i\in I} \mathrm{Sp}(\mathcal{C}_i), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_\star(\prod_{i\in I} \mathcal{C}_i, \mathcal{D})$.*

The subscript $\star$ indicates that we are considering reduced functors which preserve sequential colimits. There are two ingredients with relatively high investment needed for the proof of the above result. First we will need first derivatives, which is done in §2.3.1. Second we will need to define the suspension functor, which is done in §2.3.2. Assuming the material developed in those subsubsections, the only missing ingredient before proving the above result is the following neat lemma.

**Lemma 2.3.2.** *(Lemma 6.2.3.26. in [21]) Let $\mathcal{C}$ be a pointed differentiable category with finite colimits, and let $\mathcal{E} \subset \mathrm{Sp}(\mathcal{C})$ be a stable subcategory which contains the image of $\Sigma^\infty_{\mathcal{C}} : \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$. If $\mathcal{E}$ is closed under sequential colimits, then $\mathcal{E} = \mathrm{Sp}(\mathcal{C})$.*

*Proof.* It isn't hard to explicitly observe that the derivative of $\mathrm{Id}_{\mathcal{C}}$ can be taken to be $\mathrm{Id}_{\mathrm{Sp}(\mathcal{C})}$. Now applying the adjunction $\Sigma^\infty \dashv \Omega^\infty$ to the this statement yields that the counit map $\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}} \to \mathrm{Id}_{\mathrm{Sp}(\mathcal{C})}$ is equivalent to the map $\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}} \to P_1(\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}})$. In other words, the identity is the 1-excisive approximation to $\Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}}$, which implies the following by an explicit computation of $P_1$:

$$\mathrm{Id}_{\mathrm{Sp}(\mathcal{C})} \simeq \varinjlim_n \Omega^n_{\mathrm{Sp}(\mathcal{C})} \circ \Sigma^\infty_{\mathcal{C}} \circ \Omega^\infty_{\mathcal{C}} \circ \Sigma^n_{\mathrm{Sp}(\mathcal{C})}.$$

This means that each spectrum object $X$ can be expressed as $\varinjlim_n \Omega^n_{\mathrm{Sp}(\mathcal{C})} \circ \Sigma^\infty_{\mathcal{C}}(X(S^n))$. As $\mathcal{E}$ contains the essential image of $\Sigma^\infty$, is closed under $\Omega^n_{\mathrm{Sp}(\mathcal{C})}$ (by definition of being a stable subcategory), and is closed under sequential limit, we may conclude that $\mathcal{E}$ must contain $X$. This proves the claim. □

We may now move on to the proof of the main result.

*Proof.* We first prove that precomposition by $\Sigma^\infty_{\mathcal{C}_i}$ induces an equivalence $\phi : \mathrm{Exc}_\star(\prod_{i\in I} \mathrm{Sp}(\mathcal{C}_i), \mathcal{D}) \to \mathrm{Exc}_\star(\prod_{i\in I} \mathcal{C}_i, \mathcal{D})$. In calling this map $\phi$ we are mildly abusing notation, but this shouldn't cause any confusion. We can work one variable at a time, thus reducing to proving that $(\Sigma^\infty_{\mathcal{C}})^* = \phi : \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Exc}_\star(\mathcal{C}, \mathcal{D})$ is an equivalence. We can write this functor as the composition

$$\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xrightarrow{\phi} \mathrm{Fun}_\star(\mathcal{C}, \mathcal{D})$$

followed by an appropriate corestriction. Each of the functors above are right adjoint, and so the following composition of left adjoin adjoints assemble into an adjunction for the composite

$$\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xleftarrow{P_1} \mathrm{Fun}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \xleftarrow{\Omega^\infty} \mathrm{Fun}_\star(\mathcal{C}, \mathcal{D}).$$

This functor can be restricted, so that the whole adjunction restricts to

$$\begin{tikzcd} \mathrm{Exc}_\star(\mathcal{C}, \mathcal{D}) \arrow[rr, bend right, "\psi"'] \arrow[rr, phantom, "\top"] & & \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D}) \arrow[ll, bend right, "\phi"'] \end{tikzcd}.$$

Our goal is to promote this adjunction to an adjoint equivalence. Fix some 1-excisive reduced functor preserving sequential colimits $F$. Notice that $\psi(F) = \Omega^\infty \circ \partial F$, as can be seen by inspecting the proof of proposition 2.3.5. Our first step will be showing that $\psi$ is fully faithful, which can be done by showing that the unit map $\eta : F \to \phi \circ \psi(F) = \Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}$ is an equivalence. This can clearly be verified objectwise, so we fix an object $C$ and try to prove that $\eta_C : F(C) \to \phi \circ \psi(F) = \Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}}(C)$ is an equivalence.
We introduce some notation, namely for $T$ a singleton we abbreviate $L^T_{\mathcal{C}}$ (defined in corollary 2.3.7) to $L_{\mathcal{C}}$ and similarly for $L^T_{\mathcal{D}}$. Notice these functors are in fact 1-excisive approximation. In definition 2.3.9 we defined functor $F^+$ for any surjection $q : S \to T$, which in the case where $q : S \to T$ is just the identity of a singleton is simply post-composition by $F$. Also recall from subsubsection 2.3.2 that we have a functor $\overline{\Sigma^\infty_{\mathcal{C}}} : \mathcal{C} \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$, which is a left adjoint to evaluating at $S^0$. Notice that $\Sigma^\infty_{\mathcal{C}} \simeq L_{\mathcal{C}} \circ \overline{\Sigma^\infty_{\mathcal{C}}}$, in particular we have a map $\overline{\Sigma^\infty_{\mathcal{C}}} \to \Sigma^\infty_{\mathcal{C}}$ which is an equivalence after applying $L_{\mathcal{C}}$. Using proposition 2.3.11, we can compute the derivative of $F$ at $\Sigma^\infty_{\mathcal{C}} C$ explicitly as

$$\partial F(\Sigma^\infty_{\mathcal{C}} C) \simeq L_{\mathcal{D}} F^+(\Sigma^\infty_{\mathcal{C}} C).$$

Now post-composing with the map induced by $\overline{\Sigma^\infty_{\mathcal{C}}} \to \Sigma^\infty_{\mathcal{C}}$, we obtain a map $L_{\mathcal{D}} F^+(\overline{\Sigma^\infty_{\mathcal{C}}}) \to L_{\mathcal{D}} F^+(\Sigma^\infty_{\mathcal{C}} C)$, which is an equivalence by 2.3.12.
By remark2.3.8, we know how to compute $L_{\mathcal{D}} F^+(\overline{\Sigma^\infty_{\mathcal{C}}} C)$. At a specific object $X$ we have

$$L_{\mathcal{D}} F^+(\overline{\Sigma^\infty_{\mathcal{C}}} C)(X) \simeq \varinjlim_n \Omega^n_{\mathcal{D}}(F^+(\overline{\Sigma^\infty_{\mathcal{C}}}(C))(\Sigma^n X)).$$

In particular,for $X = S^0$, recalling proposition 1.2.6 and using that $\overline{\Sigma^\infty_{\mathcal{C}}}$ is a left adjoint, so commutes with $\Sigma^n$, we get

$$\Omega^\infty_{\mathcal{D}} \circ \partial F \circ \Sigma^\infty_{\mathcal{C}} \simeq \varinjlim_n \Omega^n_{\mathcal{D}} \circ F(\Sigma^n_{\mathcal{C}} C) \simeq P_1 F(C).$$

Under this equivalence, the map $\eta_F$ is the natural map $F \to P_1(F)$, which is an equivalence as $F$ is 1-excisive. Now, to show that $\phi, \psi$ are mutually inverse, it suffices by a sufficiently simple argument to show that $\phi$ is conservative.

Take a morphism $\beta : f \to g$ be a morphism in $\mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C}), \mathcal{D})$ such that $\phi(\beta)$ is an equivalence, we will show that $\beta$ is an equivalence. Recall that $\phi(\beta)$ is a natural transformation $f \circ \Sigma^\infty_{\mathcal{C}} \to g \circ \Sigma^\infty_{\mathcal{C}}$. We may assume that $\mathcal{D}$ is stable by invoking proposition 1.4.2.22. in [21], which says that post composing with $\Omega^\infty$ gives an equivalence $\mathrm{Exc}_*(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\mathcal{C}, \mathcal{D})$ is an equivalence. Our goal is now to show the category $\mathcal{E} \subset \mathcal{C}$ such that $\beta_X : f(X) \to g(X)$ is an equivalence is in fact all of $\mathcal{C}$. By assumption that $\phi(\beta)$ is an equivalence $\mathcal{E}$ contains the essential image of $\Sigma^\infty_{\mathcal{C}}$, and so by the lemma it suffices to show that $\mathcal{E}$ is closed under sequential limits, fibers and cofibers. Closure under sequential colimits follows from the fact that $f, g$ commute with sequential colimits. The fact that $\mathcal{E}$ is closed under fiber and cofiber sequences is because $f, g$ send pushouts to pullbacks, and because all categories involved are stale implies $f, g$ preserve pushouts and pullbacks, in particular fibers and cofibers. This completes the proof. □

### 2.3.1 First derivatives of functors

We now move on to the first definition of this section, the derivative of a functor $F : \mathcal{C} \to \mathcal{D}$, which will a an appropriate lift to a functor $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{D})$. This leads the way to the intuition that the stabilization of an $\infty$-category is analogous to the tangent space of a manifold, being the appropriate domain/codomain of the differential. This intuition is made formal in [2].

**Definition 2.3.3.** (6.2.1.1. in [21]) Let $\{\mathcal{C}_i\}_{i \in I}$ be a collection of categories admitting finite limits and $\mathcal{D}$ another such category. Given a pair of functors $F : \prod_I \mathcal{C}_i \to \mathcal{D}$ and $f : \prod_I \mathrm{Sp}(\mathcal{C}_i) \to \mathrm{Sp}(\mathcal{D})$, we say that a natural transformation $\alpha : F \circ \prod_I \Omega^\infty_{\mathcal{C}_i} \to \Omega^\infty_{\mathcal{D}} \circ f$ *exhibits $f$ as a derivative of $F$* if

(i) the functor $f$ is multilinear and

(ii) for any other multilinear functor $g : \prod_I \mathrm{Sp}(\mathcal{C}_i) \to \mathrm{Sp}(\mathcal{D})$, precomposition by $\alpha$ induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathrm{Sp}(\mathcal{D}))}(f, g) \to \mathrm{Map}_{\mathrm{Fun}(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathcal{D})}(F \circ \prod_I \Omega^\infty_{\mathcal{C}_i}, \Omega^\infty_{\mathcal{D}} \circ g)$$

As per usual, the fact that $f$ has a universal property means in so far as it exists, it is well defined up to equivalence. And so we are justified in writing $f := \vec{\partial} F$. In the case where $F$ is a single variable functor, we will simply write $\partial F$. Recall that the caveat that being a single variable or multivariable functor does not depend on $F$ but on how we decide to view $F$ (or more specifically, how we decompose the domain of $F$ as a product of categories).

*Remark* 2.3.4. (Remark 6.1.2.8. in [21]) It turns out that the second condition can be replaced by an equivalent, but easier to check statement. The equivalent formulation is given by::
For any other multilinear functor $G : \prod_I \mathrm{Sp}(\mathcal{C}_i) \to \mathcal{D}$, precomposition by $\alpha$ induces an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathcal{D})}(\Omega^\infty_{\mathcal{D}} \circ f, G) \to \mathrm{Map}_{\mathrm{Fun}(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathcal{D})}(F \circ \prod_I \Omega^\infty_{\mathcal{C}_i}, G)$$

This version of the statement follows from repeated application of proposition 1.4.2.22. of [21], which in our case says that composition with $\Omega^\infty_{\mathcal{D}}$ gives an equivalence $\mathrm{Exc}_*(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathrm{Sp}(\mathcal{D})) \to \mathrm{Exc}_*(\prod_I \mathrm{Sp}(\mathcal{C}_i), \mathcal{D})$.

We have the following general existence result for derivatives, which will be useful, though not wholly satisfactory as we will require derivatives in other contexts than those provided by this proposition.

**Proposition 2.3.5.** *(Proposition 6.2.1.9. in [21]) Let $\{\mathcal{C}_i\}_{i\in I}$ be a finite collection of categories with finite colimits, let $\mathcal{D}$ be a differentiable category, and $F : \prod_{i\in I} \mathcal{C}_i \to \mathcal{D}$ a functor which is reduced in each variable. Then $F$ admits a derivative $\vec{\partial} F : \prod_{i\in I} \mathrm{Sp}(\mathcal{C}_i) \to \mathrm{Sp}(\mathcal{D})$.*

*Proof.* This result follows from the alternate characterization of condition (ii) discussed above. Let $F' = P_{(1,\ldots,1)}(F \circ \prod_{i\in I} \Omega^\infty_{\mathcal{C}_i})$, where $P_{(1,\ldots,1)}$ was constructed in subsection §2.1. This functor is multilinear, so by proposition 1.4.2.2. in [21], this functor is given by $\Omega^\infty_{\mathcal{D}} \circ f$ for some multilinear $f : \prod_{i\in I} \mathrm{Sp}(\mathcal{C}_i) \to \mathrm{Sp}(\mathcal{D})$. Under this identification, the natural map $F \circ \prod_{i\in I} \Omega^\infty_{\mathcal{C}_i} \to P_{(1,\ldots,1)}(F \circ \prod_{i\in I} \Omega^\infty_{\mathcal{C}_i})$ becomes a map $\alpha : F \circ \prod_{i\in I} \Omega^\infty_{\mathcal{C}_i} \to \Omega^\infty_{\mathcal{D}} \circ f$. This map exhbits $f$ as a derivative of $F$ by the fact that as precomposition by $\alpha$ induces the equivalence on mapping spaces which exhibits $P_{(1,\ldots,1)}$ as a left adjoint. □

We will use this result when defining/proving the existence of $\Sigma^\infty$, and this is the only result of this subsubsection which we will use in §2.3.2, thus from here on out we may use the existence of this functor without fearing a circular reasoning.

It will be useful to have an explicit way to compute the derivative of a functor, this is what will occupy us for the rest of this section. The next definition we need is a certain special functor $L^T_{\mathcal{C}} : \mathrm{Fun}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ for every differentiable category $\mathcal{C}$ and every finite set $T$. We will state this definition after some preliminary results.
Recall that finite pointed space admit a smash product $\wedge : \mathcal{S}^{\mathrm{fin}}_* \times \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}^{\mathrm{fin}}_*$, which by a form of associativity gives an unambiguous functor $\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}^{\mathrm{fin}}_*$ for any finite set $T$, which we also denote by $\wedge$. By precomposition this gives a functor $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$, which one might study, in so far as spectrum objects (see definition A.0.11) and the smash product are natural objects to study. It turns out this functor can easily be corestricted to $\mathrm{Exc}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ by well known properties of this smash product. After corestriction, this map is quite interesting by the following result.

**Proposition 2.3.6.** *(6.2.1.11. in [21]) Let $T$ be a non-empty set and let $\mathcal{C}$ be a category with finite limits. Then precomposition with the smash product induces an equivalence*

$$\wedge^* : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Exc}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}).$$

*Proof.* Choose an index $s \in T$ and define the map $u : \mathcal{S}^{\mathrm{fin}}_* \to \prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*$ which is the identity on the $s$-component, and is constant equal to $S^0$ in every other coordinate. Precomposition by this map induces a functor $u^*$ which clearly has the property that $u^* \circ \wedge^* = (\wedge \circ u)^* \simeq \mathrm{Id}$. By 2 out of 3, it will suffice to show that $u^*$ is an equivalence, to conclude that $\wedge^*$ is so as well.
This follows from proposition A.0.12 and an appropriately restricted and corestricted currying adjunction infinity categories. Indeed, the category $\mathrm{Exc}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ can be identified with the $T$-fold application of Sp and the map $u^*$ with the $T\backslash\{s\}$-fold application of $\Omega^\infty$. □

From this we have the following corollary defining the desired functor $L^T_{\mathcal{C}}$.

**Corollary 2.3.7.** *(Corollary 6.2.1.12. in [21]) Let $T$ be a non-empty set and let $\mathcal{C}$ be a category with finite limits. Then precomposition with the smash product induces an fully faithful embedding*

$$\wedge^* : \mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}),$$

*which admits a left adjoint $L^T_{\mathcal{C}}$.*

*Proof.* The equivalence of the above proposition extends to a fully faithful embedding $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$. To see that this functor admits a left adjoint, it suffices to show that the inclusion $\mathrm{Exc}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathrm{Fun}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ admits a left adjoint. This is given by the functor $P_{(1,..,1)}$ constructed in subsection §2.1. □

*Remark* 2.3.8. Pondering the proof and recalling the construction of $P_{(1,\ldots,1)}$, we see that $\Omega^\infty \circ L^T_{\mathcal{C}}$ : $\mathrm{Fun}_*(\prod_{t\in T}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}) \to \mathcal{C}$ can be understood more explicitly as

$$\Omega^\infty \circ L^T_{\mathcal{C}}(F) = \varinjlim_{\vec{n}\in\mathbb{Z}^T_{\geq 0}} \Omega^{\vec{n}} F(\{S^{n_t}\}_{t\in T}) \in \mathcal{C}.$$

This calculation will be useful to obtain the desired result.

With this construction in hand, we can define the following model of the derivative of a functor.

**Definition 2.3.9.** (Construction 6.2.1.14. and a remark between proposition 6.2.1.18. and 6.2.1.19. in [21]) Let $q : S \to T$ be a surjection of non-empty finite sets. For each $t \in T$ denote by $S_t$ the fiber of $q$ over $t$. Let $\mathcal{C}_t$ be a collection of $T$-indexed categories with finite limits and let $\mathcal{D}$ be a differentiable category. For every reduced functor $F : \prod_{t\in T}\mathcal{C}_t \to \mathcal{D}$ denote by $F^+ : \prod_{t\in T}\mathrm{Fun}_*(\prod_{s\in S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t) \to \mathrm{Fun}_*(\prod_{t\in T}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{D})$ the functor obtained by postcomposition

$$\prod_{s\in S}\mathcal{S}^{\mathrm{fin}}_* \to \prod_{t\in T}\mathcal{C}_t \xrightarrow{F} \mathcal{D}.$$

We define $F'$ be the following composition

$$\prod_{t\in T}\mathrm{Sp}(\mathcal{C}_t) \to \prod_{t\in T}\mathrm{Fun}_*(\prod_{S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t) \xrightarrow{F^+} \mathrm{Fun}_*(\prod_{t\in T}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{D}) \xrightarrow{L^T_{\mathcal{D}}} \mathrm{Sp}(\mathcal{D}).$$

In order for the first map to make sense, we use the equivalence proven above of $\mathrm{Sp}(\mathcal{C})$ with $\mathrm{Exc}_*(\prod_{t\in T}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C})$. Given $X_t : \prod_{S_t}\mathcal{S}^{\mathrm{fin}}_* \to \mathcal{C}_t$ for each $t$, the above composition gives us a functor $L^T_{\mathcal{D}}F^+(\{X_t\}_{t\in T})$. Now, we use counit of the adjunction $L^T_{\mathcal{D}} \vdash \wedge^*$ which gives us a map $L^T_{\mathcal{D}} \circ \wedge^* \circ F^+(\{X_t\}_{t\in T}) \to F^+(\{X_t\}_{t\in T}$, evaluating both sides in $(S^0,\ldots,S^0)$ and using that $S^0$ is the monoidal unit and recalling various definitions gives a natural transformation $F \circ \prod_{t\in T}\Omega^\infty_{\mathcal{C}_t} \to \Omega^\infty_{\mathcal{D}} \circ F'$, which we will call $\alpha$.
Both in the notation $F'$ and $\alpha$, we suppress the dependence on $q : S \to T$, which will be justified as will show that $\alpha$ expresses $F'$ as the derivative of $F$, and in particular is well defined up to equivalence.

To show that this is indeed an appropriate model for $\vec{\partial}(F)$, we need to show it satisfies both conditions of definition 2.3.3. This is done by the following pair of propositions.

**Proposition 2.3.10.** *(Proposition 6.2.1.18. in [21]) Let $\{\mathcal{C}_t\}_{t\in T}$ be a finite collection of differentiable categories, $\mathcal{D}$ another such category and $F : \prod_{t\in T}\mathcal{C}_t \to \mathcal{D}$ a functor which is reduced in each variable and preserves sequential colimits. For every surjection $q : S \to T$, the functor $F' : \prod_{t\in T}\mathrm{Sp}(\mathcal{C}_t) \to \mathrm{Sp}(\mathcal{D})$ of the previous definition preserves countable colimits separately in each variable. In particular, $F'$ is multilinear.*

*Proof.* Multilinearity will follow from stability of the codomain, as then in $\mathrm{Sp}(\mathcal{D})$ pushouts are the same as pullbacks. We will henceforth allow ourselves to omit specifying "separately in each variable". To show that $F'$ preserves all countable colimits, we start by showing it preserves sequential colimits. For this we show that each functor in the composition defining $F'$ preserves colimits. For convenience we recall that $F'$ is given by

$$\prod_{t\in T}\mathrm{Sp}(\mathcal{C}_t) \to \prod_{t\in T}\mathrm{Fun}_*(\prod_{S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t) \xrightarrow{F^+} \mathrm{Fun}_*(\prod_{t\in T}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{D}) \xrightarrow{L^T_{\mathcal{D}}} \mathrm{Sp}(\mathcal{D}).$$

The last map preserves all colimits as it is a left adjoint, and the middle map because $F$ does. The first map does as it is the composition of an equivalence and the inclusion $\mathrm{Exc}_*(\prod_{S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t) \to \mathrm{Fun}_*(\prod_{S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t)$, which preserves sequential colimits as $\mathrm{Exc}_*(\prod_{S_t}\mathcal{S}^{\mathrm{fin}}_*,\mathcal{C}_t)$ is always closed under sequential colimits.
It now follows that $F^+$ preserves countable filtered colimits, because by finality arguments we can replace filtered diagrams $J \to \mathcal{C}$ by diagrams of the form $\mathrm{N}(A)$ where $A$ is a countable poset (see statement and proof of 5.1.3.18. in [20]). We can then easily replace $A$ by $\mathbb{Z}_{\geq 0}$ by replacing by a final subdiagram, which we are allowed to do by proposition A.0.7. Now it suffices to show that $F'$ preserves

countable coproducts and coequalizers by 4.4.3.2. in [20]. Countable coproducts will be preserved if finite coproducts are preserved because $F'$ preserves sequential colimits. Thus it will suffice to show that $F'$ is right exact.
We now fix every variable but one, i.e. we consider the functor $G : X \mapsto F'(X, \{Y_t\}_{t\in T\setminus\{t_0\}})$ where the $Y_t$ are fixed. By corollary 1.4.2.14. and 1.1.4.1., to show that $G$ is exact, it will suffice to show that $G$ maps the zero object to the zero object and that the natural map $\Sigma_{\mathcal{D}}G(X) \to G(\Sigma_{\mathcal{C}}X)$ is an equivalence. The fact that $G$ preserves 0-objects follows immediately from $F$ being reduced. To verify the other condition, we will need to take a detour to obtain a different formulation of the natural map $\Sigma_{\mathcal{D}}G(X) \to G(\Sigma_{\mathcal{C}}X)$.
Let $q : S \to T$ be the surjection appearing in the definiton of $F'$ and choose $s_0$ mapping to $t_0$. Let $U : \prod_{s\in S}\mathcal{S}_*^{\text{fin}} \to \prod_{s\in S}\mathcal{S}_*^{\text{fin}}$ be the functor which suspends the $s_0$th coordinate, and is the identity on all the other coordinates. By proposition 2.3.6 properties of the smash product, of suspension of finite spaces and because $\Sigma_{\mathrm{Sp}(\mathcal{D})}$ can be computed pointwise, we may fit $U$ into the following commutative diagram:

$$\begin{tikzcd} \mathrm{Sp}(\mathcal{D}) \arrow[r, "\Sigma_{\mathrm{Sp}(\mathcal{D})}"] \arrow[d] & \mathrm{Sp}(\mathcal{D}) \arrow[d] \\ \mathrm{Exc}_*(\prod_{s\in S}\mathcal{S}_*^{\text{fin}}, \mathcal{D}) \arrow[r, "-\circ U"] & \mathrm{Exc}_*(\prod_{s\in S}\mathcal{S}_*^{\text{fin}}, \mathcal{D}) \end{tikzcd}.$$

We now define the functor $Z : \prod_{s\in S}\mathcal{S}_*^{\text{fin}} \to \mathcal{D}$ by the following composition

$$\prod_S \mathcal{S}_*^{\text{fin}} \simeq \prod_{t\in T}\prod_{S_t}\mathcal{S}_*^{\text{fin}} \xrightarrow{\prod_{t\in T}\wedge} \prod_{t\in T}\mathcal{S}_*^{\text{fin}} \xrightarrow{X,\{Y_t\}_{t\in T\setminus\{t_0\}}} \prod_{t\in T}\mathcal{C}_t \xrightarrow{F} \mathcal{D}.$$

Now notice that the natural map $P_{(1,..,1)}(Z) \circ U \to P_{(1,..,1)}(Z \circ U)$ is an equivalence, because the map $U$ clearly satisfies the assumptions of lemma 1.2.9. At this point, one might reasonably ask why we care about this map. Chasing definitions around we see this is in fact the map we are interested in. In particular, one must notice that suspension in the category of spectrum objects corresponds to pointwise suspension, i.e.

$$\Sigma_{\mathrm{Sp}(\mathcal{C})}X \simeq X \circ \Sigma_{\mathcal{S}_*^{\text{fin}}}.$$

This concludes the proof. $\square$

**Proposition 2.3.11.** *(Proposition 6.2.1.19. in [21]) Let $\{\mathcal{C}_t\}_{t\in T}$ be a finite collection of differentiable categories, $\mathcal{D}$ another such category and $F : \prod_{t\in T}\mathcal{C}_t \to \mathcal{D}$ a functor which is reduced in each variable and preserves sequential colimits. For every surjection $q : S \to T$, the natural transformation $\alpha : F \circ \prod_{t\in T}\Omega^\infty_{\mathcal{C}_t} \to \Omega^\infty_{\mathcal{D}} \circ F'$ from the above definition exhibits $F'$ as a derivative of $F$.*

*Proof.* We have a natural map $\alpha : F \circ \prod_{t\in T}\Omega^\infty_{\mathcal{C}_t} \to \Omega^\infty_{\mathcal{D}} \circ F'$. Since $F'$ is multilinear, the same holds for $\Omega^\infty \circ F'$, because $P_{(1,...,1)}$ is left adjoint to the inclusion of $(1, ..., 1)$-excisive functors, we get that $\alpha$-factors as

$$F \circ \prod_{t\in T}\Omega^\infty_{\mathcal{C}_t} \xrightarrow{\alpha'} P_{(1,...,1)}(F \circ \prod_{t\in T}\Omega^\infty_{\mathcal{C}_t}) \xrightarrow{\alpha''} \Omega^\infty_{\mathcal{D}} \circ F'.$$

Using the universal property of $P_{(1,...,1)}$ we see that $\alpha'$ is an equivalence, so that it suffices to show that $\alpha''$ is an equivalence. We show the isomorphism pointwise, and leave it to the reader to verify that the equivalence we show is indeed the map $\alpha''$. Fix $\{X_t\}_{t\in T}$ some collection of objects in $\mathrm{Sp}(\mathcal{C})$. For $\vec{n} \in \mathbb{Z}^S_{\geq 0}$, denote by $\vec{n}_t$ the restriction of $\vec{n}$ to $S_t$, denoting by $|\vec{n}|$ the sum of the coordinates, this gives a map $\sigma : \mathbb{Z}^S_{\geq 0} \to \mathbb{Z}^T_{\geq 0}$ by sending $\vec{n}$ to $\{|\vec{n}_t|\}_{t\in T}$, we claim that this map is cofinal. This can be seen using theorem A.0.10, as for each $\vec{p} \in \mathbb{Z}^T_{\geq 0}$, the category $\mathbb{Z}^S_{\geq 0} \times_{\mathbb{Z}^T_{\geq 0}} \mathbb{Z}^T_{\geq 0,\vec{p}/}$ has an initial object given by $\vec{p} \to \sigma(\vec{k})$, where $\vec{k} = \{\{k_s\}_{s\in S_t}\}_{t\in T}$ and we choose an $s_t \in S_t$ for each $t$, set $k_{s_t} = p_t$ and $k_s = 0$ otherwise. So in paritcular $\mathbb{Z}^S_{\geq 0} \times_{\mathbb{Z}^T_{\geq 0}} \mathbb{Z}^T_{\geq 0,\vec{p}/}$ is contractible. The construction of this initial object involved some choice, but in fact any choice yields objects which are equivalent up to contractible choice.

Now that this is said, the result follows upon pondering the following chain of equivalences, which we obtain courtesy of remark 2.3.8 and all of the definitions involved:

$$
\begin{aligned}
& P_{(1,\dots,1)}(F \circ \Omega^\infty_{\mathcal{C}})(\{X_t\}_{t\in T}) \\
& \simeq \varinjlim_{\vec{m}\in\mathbb{Z}^T_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} \circ (F \circ \Omega^\infty_{\mathcal{C}}) \circ \prod_{t\in T} \Sigma^{m_t}_{\mathrm{Sp}(\mathcal{C})}(X_t) \\
& \simeq \varinjlim_{\vec{m}\in\mathbb{Z}^T_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} \circ (F \circ \Omega^\infty_{\mathcal{C}}) \circ \prod_{t\in T} (X_t \circ \Sigma^{m_t}_{\mathcal{S}^{\mathrm{fin}}_*}) \\
\simeq \varinjlim_{\vec{n}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{n}|}_{\mathcal{D}} \circ (F(\Sigma^{n_t} S^0)) \simeq \Omega^\infty_{\mathcal{D}} \circ L^T_{\mathcal{D}}(F^+(S^{n_t})) \simeq \Omega^\infty_{\mathcal{D}} \circ F'(\{X_t\}).
\end{aligned}
$$

□

When applying the above result in the proof of the main result of this section, we will need the following result. When we say $F$-equivalence, we mean a map whose image is an equivalence.

**Proposition 2.3.12.** *(Proposition 6.2.1.20. in [21]) Let $\{\mathcal{C}_t\}_{t\in T}$ be a finite collection of differentiable categories, $\mathcal{D}$ another such category and $F : \prod_{t\in T} \mathcal{C}_t \to \mathcal{D}$ a functor which is reduced in each variable and preserves sequential colimits. For every surjection $q : S \to T$, the functor $F^+ : \prod_{t\in T} \mathrm{Fun}_*(\prod_{s\in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t) \to \mathrm{Fun}_*(\prod_{t\in T} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{D})$ defined in the above definition carries $\prod_{t\in T} L^{S_t}_{\mathcal{C}_t}$-equivalences to $L^S_{\mathcal{D}}$-equivalences.*

*Proof.* Suppose we are given maps $\{\alpha_t : X_t \to Y_t\}_{t\in T}$ which seen as a single map in $\prod_{t\in T} \mathrm{Fun}_*(\prod_{s\in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t)$ is a $\prod_{t\in T} L^{S_t}_{\mathcal{C}_t}$-equivalences. To help clear up notation, denote by $P_{\vec{1}}$ the left adjoint to the inclusion $\mathrm{Fun}_*(\prod_{s\in S} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{D}) \to \mathrm{Exc}_*(\prod_{s\in S} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{D})$ and denote by $P^t_{\vec{1}}$ the left adjoint to the inclusion $\mathrm{Fun}_*(\prod_{s\in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t) \to \mathrm{Exc}_*(\prod_{s\in S_t} \mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}_t)$. Using this notation and recalling the definition of $L^{S_t}_{\mathcal{C}_t}$ we see that our condition on the $\{\alpha_t\}$ is equivalent to the requirement that $P_{\vec{1}}(\alpha_t)$ is an equivalence. And similarly, we may restate what we want to show as showing that the induced map $P_{\vec{1}}F^+(\{X_t\}) \to P_{\vec{1}}F^+(\{Y_t\})$ is an equivalence.
Using proposition 1.2.6, the definition of $F^+$ and the fact that an equivalence of functors can be verified pointwise, we reduce to showing that

$$
\gamma : \varinjlim_{\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|} F(\{X_t(\{\Sigma^{m_s} K_s\}_{s\in S_t})\}_{t\in T}) \to \varinjlim_{\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|} F(\{Y_t(\{\Sigma^{m_s} K_s\}_{s\in S_t})\}_{t\in T})
$$

is an equivalence, where $K_s$ is some fixed collection of finite spaces. For $\vec{n} \in \mathbb{Z}^S_{\geq 0}$, denote by $\vec{n}_t$ the restriction to $S_t$. We can fit $\gamma$ in the following commutative diagram

$$
\begin{tikzcd}
\varinjlim_{\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{X_t(\{\Sigma^{m_s} K_s\}_{s\in S_t})\}_{t\in T}) \arrow[r, "\gamma"] \arrow[d] & \varinjlim_{\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{Y_t(\{\Sigma^{m_s} K_s\}_{s\in S_t})\}_{t\in T}) \arrow[d] \\
\varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{\Omega^{\vec{n}_t}_{\mathcal{C}_t} X_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T}) \arrow[r, "\gamma'"] \arrow[d] & \varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|}_{\mathcal{D}} F(\{\Omega^{\vec{n}_t}_{\mathcal{C}_t} Y_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T}) \arrow[d] \\
\varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|+|\vec{n}|}_{\mathcal{D}} F(\{X_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T}) \arrow[r, "\gamma''"] & \varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \Omega^{|\vec{m}|+|\vec{n}|}_{\mathcal{D}} F(\{Y_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T})
\end{tikzcd}
.
$$

A similar cofinality argument to the one made explicit in the proof of the previous proposition shows that the vertical compositions are equivalences. So that the map $\gamma$ we are interested in is a retract of $\gamma'$, so it suffices to show that $\gamma'$ is an equivalence. Now because $F$ commutes with sequential colimits by assumption and $\Omega_{\mathcal{D}}$ because $\mathcal{D}$ is differentiable, it suffices to show that the maps

$$
\varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \{\Omega^{\vec{n}_t}_{\mathcal{C}_t} X_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T} \to \varinjlim_{\vec{n},\vec{m}\in\mathbb{Z}^S_{\geq 0}} \{Y_t(\{\Sigma^{m_s+n_s} K_s\}_{s\in S_t})\}_{t\in T}
$$

are equivalences. But this follows by assumption on $\alpha_t$ and proposition 1.2.6. □

### 2.3.2 The suspension functor

In this subsubsection we define a highly expected functor, a left adjoint to $\Omega^\infty$. When it exists, we'll call this the infinite suspension, and denote it by $\Sigma^\infty_+$, omitting the $+$ subscript if $\mathcal{C}$ is pointed. Just like as for $\Omega^\infty$, we will sometimes use the category whose suspension functor it is as a subscript, i.e. the adjoint to the functor $\Omega^\infty_\mathcal{C} : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ might be denoted by $\Sigma^\infty_{+,\mathcal{C}}$ or $\Sigma^\infty_\mathcal{C}$.

We will prove the existence of $\Sigma^\infty_+$ only in a case of interest to us, though there are others (see for example proposition 1.4.4.4. in [21] for the case of $\mathcal{C}$ a presentable category).

**Proposition 2.3.13.** *(6.2.3.16. and 6.2.3.19. in [21]) Let $\mathcal{C}$ be a category with finite colimits and a final object and let $\theta : \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \to \mathcal{C}$ denote evaluation at $S^0$. Then $\theta$ admits a left adjoint $\overline{\Sigma^\infty_\mathcal{C}} : \mathcal{C} \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$.*

*If in addition $\mathcal{C}$ is differentiable, then $\Sigma^\infty_\mathcal{C} = P_1(\overline{\Sigma^\infty_\mathcal{C}})$ is a left adjoint to $\Omega^\infty_\mathcal{C}$.*

*Proof.* The functor $\Omega^\infty$ factors as $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C}) \xrightarrow{\theta} \mathcal{C}$, so if we find a left adjoint for $\theta$, a left adjoint to $\Omega^\infty$ is given by post composing this with the left adjoint to the other map, which is $P_1$ by theorem 1.2.1. So it suffices to prove the first half of the above statement. We will do this by explicitly constructing $\overline{\Sigma^\infty}$ .

Let $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{Fin}}, \mathcal{C})$ be the subcategory of $\mathrm{Fun}(\mathcal{S}^{\mathrm{Fin}}, \mathcal{C})$ consisting of the right exact functors, i.e. the maps which are left Kan extensions of their restriction to the one point space $*$. Applying proposition A.0.5 we get that the restriction $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{Fin}}, \mathcal{C}) \to \mathrm{Fun}(*, \mathcal{C})$ is a trivial fibration. This map can clearly be identified with the map $\mathrm{Fun}^{\mathrm{Rex}}(\mathcal{S}^{\mathrm{Fin}}, \mathcal{C}) \to \mathcal{C}$ given by evaluation at a one point space $*$. Choose a section of this map, which we denote by $f_\bullet$, mapping an object $C$ to $f_C$. We turn this into a functor to $\mathcal{S}^{\mathrm{fin}}_*$ by mapping a pointed space $K$ to $f_C(K) \sqcup_C *$ where $*$ is a final object of $\mathcal{C}$ and the map $C \to f_C(K)$ is given by applying $f_C$ to the unique map to the base point $* \to K$. We denote the functor $C \to \mathrm{Fun}_*(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})$ by $\overline{\Sigma^\infty}$.

For each $C \in \mathcal{C}$, we have a canonical equivalence $\theta(\overline{\Sigma^\infty}) = f_C(S^0) \sqcup * \simeq C \sqcup_C *$. This equivalence can easily be used to construct a map $\eta : \mathrm{Id} \to \theta \circ \overline{\Sigma^\infty}$. We will show that this $\eta$ is the unit of an adjunction. For this, fix a $C \in \mathcal{C}$ and a reduced functor $g : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{C}$, we want to show that the following composition is an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, C)}(f^+_C, g) \to \mathrm{Map}_\mathcal{C}(f^+_C(S^0), g(S^0)) \to \mathrm{Map}_\mathcal{C}(C, g(S^0)).$$

The first map is given by applying $\theta$ and the second is precomposition by $\eta$.

Now notice that if $\underline{A} : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{C}$ is the constant functor equal to $A$, then $\underline{A}$ is left Kan extension of its restriction to $* \in \mathcal{S}^{\mathrm{fin}}_*$ which is a zero object. In particular, the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(\underline{A}, g)$ is contractible by the left Kan extension adjunction and because $g$ is reduced. Denoting by $U : \mathcal{S}^{\mathrm{fin}}_* \to \mathcal{S}^{\mathrm{Fin}}$ the forgetful functor we can consider the following pushout:

$$\begin{tikzcd}
\underline{C} \arrow[r, "!"] \arrow[d] & \underline{*} \arrow[d] \\
f_C \circ U \arrow[r] & f^+_C
\end{tikzcd}$$

and apply $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(-, g)$, which because the top row consists of contractible object (and thus is an equivalence) and because equivalences are stable under pushout yields an equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(f^+_C, g) \to \mathrm{Map}_{Fun(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(f_C \circ U, g).$$

Now using that $U$ is left adjoint to the functor $(-)_+$ which freely adjoins a base point, we may replace $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(f^+_C, g)$ by $\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(f, g \circ (-)_+)$. Chasing around some definitions, we see the map we want to show is an equivalence is the map obtained by evaluation at $*$

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}}_*, \mathcal{C})}(f_C, g \circ (-)_+) \to \mathrm{Map}_\mathcal{C}(C, g \circ (-)_+(*)).$$

But this in fact follows from the adjunction definition that $f_C$ is a left Kan extension of its restriction to $*$. This proves the proposition. $\square$

## 2.4 The definition

With all of these results, we are finally ready to define the derivative of a functor. Let $\mathcal{C}$ be a pointed, differentiable category with finite colimits and let $\mathcal{D}$ be a pointed differentiable category. These are the necessary assumptions so that we can freely use all the results developed thus far, which will in particular allow us to define the $n$th derivative $\partial_n F \in \mathrm{Exc}_\star(\mathrm{Sp}(\mathcal{C})^n, \mathrm{Sp}(\mathcal{D}))$.
First, by theorem 1.2.1, we have $n$-excisive approximations $P_n F$, which in classical calculus correspond to degree $n$ Taylor polynomials. Substracting the degree $n-1$ approximation from the degree $n$ approximation yields a term of the form $\frac{f^{(n)}(a)}{n!}x^n$, which in our case is done by taking the fiber of the natural map $P_n F \to P_{n-1}F$, which we denote by $D_n F$. By theorem 1.3.2 (and the fact that $\Omega_{\mathcal{D}}$ preserves finite limits), this fiber is $n$-homogeneous. Now taking the $n$th cross effect yields a symmetric in $n$-variables linear functor by theorem 2.2.1. In the classical picture, the functor $\mathrm{cr}_{(n)}(D_n F)$ corresponds to $f^{(n)}(a)x_1 \cdots x_n$. The final step is to choose a specific $a$, i.e. to point our discussion. This is done by theorem 2.3.1, which fits $\mathrm{cr}_{(n)}(D_n F)$ into the following commutative diagram

$$\begin{tikzcd} \mathrm{Sp}(\mathcal{C})^n \arrow[r] & \mathrm{Sp}(\mathcal{D}) \arrow[d, "\Omega^\infty_{\mathcal{D}}"] \\ \mathcal{C}^n \arrow[u, "\Sigma^\infty_{\mathcal{C}}"] \arrow[r, "\mathrm{cr}_{(n)}(D_n F)"] & \mathcal{D} \end{tikzcd} .$$

We define the top map to be $\partial_n F$, which is a symmetric multilinear functor, and hopefully the above paragraph is enough to convince the reader that this is a reasonable definition.

Following the introduction to section 6.3.3 in [21], we can observe by multilinearity (or more specifically by corollary 1.4.4.6 in [21]) of $\partial_n F$ that in the case $\mathcal{C}$ is a category such that $\mathrm{Sp}(\mathcal{C})$ is the category of spectra (in spaces), then $\partial_n F$ is fully determined by the image of $(\mathbb{S}, \mathbb{S}, ..., \mathbb{S}) \in \mathrm{Sp}$. This in particular implies that in this case, the derivatives of a functor assemble into a symmetric sequence of spectra.

In part II of this project, we will discuss some of the prerequisites need to understand the description of the symmetric sequence obtained from $\mathrm{Id} : \mathcal{S} \to \mathcal{S}$ given by Ching in [6]. We leave it to the reader to recall that all the necessary assumptions to apply Goodwillie calculus are satisfied by the identity functor on spaces.

# 3 Various tangents to understand Ching's article

In this section, we gather a collection of tangents which together allow a reader with working knowledge of algebraic topology (say at the level of Hatcher's eponymous book [14] and able to work with spectral sequences coming from an exact couple) and with sufficient categorical maturity to read Ching's paper [6] with relative ease. We do not claim that what we write here removes any work from reading the paper [6], simply that we discuss enough prerequisites to shed some light on some concepts used, thus turning various black boxes gray. Throughout this part of the project, whenever we say "Ching's paper" without specifying further, we mean [6].

When we do enriched category theory, will always implicitly assume the base $\mathcal{V}$ to be a closed symmetric monoidal category which is complete and cocomplete. For ease of convenience, we will call such categories Bénabou cosmoi, as suggested by [25]. The main examples to keep in mind are one's favourite category of spaces and/or of spectra. The enriched categories we will be working with are those which are used in Ching's paper, whose definition we recall here.

**Definition 3.0.1.** (1.10 in [6]) Let $(\mathcal{V}, \wedge, I)$ be a Bénabou cosmos, we call $(\mathcal{M}, \otimes, \mathbf{1}, d)$ a *symmetric monoidal $\mathcal{V}$-category* if $(\mathcal{M}, \otimes, \mathbf{1})$ is itself symmetric monoidal, which is also:

(i) enriched,

(ii) tensored (which unless specified otherwise we denote by $(-) \cdot \bullet : \mathcal{V} \times \mathcal{M} \to \mathcal{M}$) over $\mathcal{V}$,

(iii) and cotensored (which unless specified otherwise denote by $\bullet^{(-)} : \mathcal{V}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$) over $\mathcal{V}$.

To be clear, in the tensoring and in the cotensoring, the $\bullet$ is a placeholder for the object coming from $\mathcal{M}$ and the $(-)$ is a placeholder for the object coming from $\mathcal{V}$.
The final piece of data is the map $d$, which is required to be a natural transformation $d : (X \wedge Y) \cdot (C \otimes D) \to (X \cdot C) \otimes (Y \cdot D)$, with $X, Y \in \mathcal{V}$ and $C, D \in \mathcal{M}$ satisfying some compatibility axioms which we do not recall.

Much of what we do can be done in greater generality than the above context, and the interested reader may refer to our sources to see which part of the above structure is really necessary. We will always work with categories as in the above definition essentially for the sake of homogeneity.

## 3.1 (Co)end calculus

In this subsection we give a brief introduction to (co)end calculus, following [19]. The main goal for us, is to give some familiarity with this somewhat more specialized notion of category theory. More specifically we will use the coend to discuss geometric realization of simplicial sets, which we hope will give the intution of the connection between coends and geometric realization. This intution will also be reinforced by understanding of how to tensor functors, which we will also discuss.

We first simply state the necessary definition and basic results required to work with (co)ends.

**Definition 3.1.1.** (Definition 1.2, 1.5 and 1.6 in [19])

(i) Given two functors $P, Q : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ a *dinatural transformation* between them, which we depict as an arrow $\alpha : P \xrightarrow{\cdot\cdot} Q$ is a collection of morphisms $\alpha_c : P(c,c) \to Q(c,c)$ such that given any morphism $f : c \to c'$ in $\mathcal{C}$ the following diagram commutes

$$\begin{tikzcd}
 & P(c',c') \arrow[r, "\alpha_{c'}"] & Q(c',c') \arrow[dr, "Q(f{,}c')"] & \\
P(c',c) \arrow[ur, "P(c'{,}f)"] \arrow[dr, "P(f{,}c)"'] & & & Q(c,c') \\
 & P(c,c) \arrow[r, "\alpha_c"'] & Q(c,c) \arrow[ur, "Q(c{,}f)"'] &
\end{tikzcd} .$$

(ii) Let $P : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. A *wedge* is a dinatural transformation $\underline{d} \xrightarrow{..} P$ where $\underline{d}$ is the constant functor which on objects sends everything to $d$ in $\mathcal{D}$ and sends every morphism to the appropriate identity. Dually a *cowedge* is a dinatural transformation to a functor of the forme $\underline{d}$.

(iii) The *end* of a functor $P : \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, denoted by $\int_{c\in\mathcal{C}} P$ is a terminal wedge. Dually a *coend* is an initial cowedge and is denoted by $\int^{c\in\mathcal{C}} P$.

Note that we might allow ourselves to denote (co)ends in similar ways, but potentially suppressing some information, for example by $\int_{\mathcal{C}} P$ or $\int_c P$. We also observe that (co)ends need not exist in general, and we will be tacitly assuming their existence whenever we use the notation. At this point we may introduce a first example of a (co)end.

*Example* 3.1.2. Consider an object in the category $G - \mathrm{Set} - G$, i.e. a set with a left and a right $G$-action. These can be viewed as functors $BG \times BG^{\mathrm{op}} \to \mathrm{Set}$, where $BG$ is the one object category corresponding to the group $G$. Given this categorical description of a $G - \mathrm{Set} - G$, it is natural to wonder what the end and coend of such an object is.
One can verify that the end of a $G - \mathrm{Set} - G$ $X$ is the subset where the two actions agree and the coend is the quotient of $X$ by the relation $gx \sim xg$.

We now state some fundamental results about coends without proof, as the proofs are just verification of universal properties. For the first of these results, we need the definition of $\mathrm{TW}(\mathcal{C})$, the category of twisted arrows in $\mathcal{C}$ whose objects are morphisms in $\mathcal{C}$ and whose morphisms from $a \to b$ to $c \to d$ are commutative squares of the form

$$\begin{tikzcd} a \arrow[r] & b \arrow[d] \\ c \arrow[u] \arrow[r] & d \end{tikzcd}.$$

We also need to notice that there is an obvious functor $\mathrm{TW}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

**Proposition 3.1.3.** *(Section 1.5. in [19]) The end $\int_{c\in\mathcal{C}} P$ can be realized as the limit of $P$ restricted to* $\mathrm{TW}(\mathcal{C})$*. Dually, a coend can be realized as the colimit of the same restriction.*

This characterization is useful in order to painlessly transfer results about (co)limits to (co)ends. Another useful characterization as a (co)limit is given by the following result, which can have the added advantage of being computationally useful.

**Proposition 3.1.4.** *(Remark 1.23. in [19]) The end $\int_{c\in\mathcal{C}} P$ can be realized as*

$$\mathrm{eq}\Big(\ \prod_{c\in\mathcal{C}} F(c,c) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \prod_{(c\to c')\in\mathcal{C}} F(c,c')\ \Big).$$

*Dually the coend $\int^{c\in\mathcal{C}}$ can be realized as*

$$\mathrm{coeq}\Big(\ \sqcup_{c\in\mathcal{C}} F(c,c) \underset{F_*}{\overset{F^*}{\rightrightarrows}} \sqcup_{(c\to c')\in\mathcal{C}} F(c,c')\ \Big).$$

We can use the above characterization to compute another natural end. This example may also convince the reader that this construction is more than abstract nonsense (or at the very least that it is relevant abstract nonsense for readers who already care about abstract nonsense).

*Example* 3.1.5. (Theorem 1.29. in [19]) Consider $F, G : \mathcal{C} \to \mathcal{D}$ two functors, we then have a natural functor $\mathrm{Hom}(F(-), G(-)) : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. The end of this functor is $\mathrm{Nat}(F, G)$.

This example gives a different intution for (co)ends than the one we wish to develop as assembling local information $(\mathrm{Hom}(F(c), G(c')))$ into global information $(\mathrm{Nat}(F, G))$. This perspective has a suprising realization in physics, for which we invite the reader to consult [9].

The following results give us some of the fundamental computational tools for dealing with (co)ends.

**Proposition 3.1.6.** *(Theorem 1.26. in [19]) Continuous functors $\mathcal{D} \to \mathcal{E}$ preserve ends. Dually cocontinuous functors preserve coends.*

**Proposition 3.1.7.** *(Remark 1.16. in [19])*

*(i) (The freshman's dream for (co)ends) For a natural transformation $\eta : F \Rightarrow G$ between functors with domain $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. There is an induced map $\int_{c\in\mathcal{C}} \eta : \int_{c\in\mathcal{C}} F \to \int_{c\in\mathcal{C}} G$. Given two composable natural transformations between such functors $\sigma, \tau$ we have the relationship*

$$\int_{c\in\mathcal{C}} \tau \circ \sigma = \int_{c\in\mathcal{C}} \tau \circ \int_{c\in\mathcal{C}} \sigma.$$

*The same formula holds for coends.*

*(ii) (Fubini's theorem for (co)ends) Given a functor $F : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ we have three ways of forming the (co)end, these all turn out to be equal, i.e.*

$$\int_{\mathcal{C}\times\mathcal{D}} F = \int_{\mathcal{C}} \int_{\mathcal{D}} F = \int_{\mathcal{D}} \int_{\mathcal{C}} F.$$

With these basics in hand, we give an application to the existence of Kan extension, which we prove as the method is emblematic of the kind of abstract non-sense used when dealing with (co)ends. For $X$ a set and $c$ an object in a cocomplete category, we denote by $X \cdot c$ the natural tensoring over sets.

**Theorem 3.1.8.** *(Section 2.1. in [19]) Let $\mathcal{E}$ be a cocomplete category, $F : \mathcal{C} \to \mathcal{D}$ a functor whose domain is small. This induces a map $F^* : \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$ by precomposition, which admits a left adjoint $\mathrm{Lan}_F : \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})$. This left adjoint is given by mapping $G : \mathcal{C} \to \mathcal{E}$ to*

$$\int^{c} \mathrm{Hom}_{\mathcal{D}}(Fc, -) \cdot Gc.$$

*Proof.* The proof follows by showing the adjunction isomorphism on Hom-sets by abstract nonsense, using the tools stated above starting with the end-description of $\mathrm{Nat}(F, G)$:

$$\mathrm{Nat}(\int^{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(Fc, -) \cdot Gc, H) \cong \int_{d\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\int^{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(Fc, d) \cdot Gc, Hd).$$

Then we can use the fact that the Hom functors sends coends in the first vatiable to ends, because of the corresponding result that it sends colimits to limits:

$$\int_{d\in\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(\int^{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(Fc, d) \cdot Gc, Hd) \cong \int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cdot Gc, Hd).$$

Now by definition of tensors we get

$$\int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathcal{D}}(Fc, d) \cdot Gc, Hd) \cong \int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_{\mathcal{D}}(Fc, d), \mathrm{Hom}_{\mathcal{E}}(Gc, Hd)).$$

Next we use the Fubini theorem for ends and the end description of natural transformation to get

$$\int_{d\in\mathcal{D}} \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_{\mathcal{D}}(Fc, d), \mathrm{Hom}_{\mathcal{E}}(Gc, Hd)) \cong \int_{c\in\mathcal{C}} \mathrm{Nat}(\mathrm{Hom}_{\mathcal{D}}(Fc, -), \mathrm{Hom}_{\mathcal{E}}(Gc, H-)).$$

We then apply the Yoneda lemma to see that

$$\int_{c\in\mathcal{C}} \mathrm{Nat}(\mathrm{Hom}_{\mathcal{D}}(Fc, -), \mathrm{Hom}_{\mathcal{E}}(Gc, H-)) \cong \int_{c\in\mathcal{C}} \mathrm{Hom}_{\mathcal{E}}(Gc, HFc).$$

And by a final application of the end characterization of natural transformation this is $\mathrm{Nat}(G, HF) = \mathrm{Nat}(G, F^*H)$. This concludes the proof. □

This can be used to prove the "nerve-realization paradigm", which is the following proposition. We denote the Yoneda embedding by よ $: \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ (the character よ is read as Yo).

**Theorem 3.1.9.** *(proposition 3.2. in [19]) Let $\phi : \mathcal{C} \to \mathcal{D}$ be a functor from a small to a cocomplete category, and let $R_\phi = \mathrm{Lan}_{よ}\phi : \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \to \mathcal{D}$ denote the left Kan extension of $\phi$ along the Yoneda embedding. This functor is a left adjoint.*

*Proof.* For the duration of this proof we denote the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$ by $\mathbf{C}$. Because $\mathcal{D}$ is cocomplete it is tensored over sets so that we can use the previous result describing left Kan extensions. So the result follows by the following chain of abstract nonsense isomoprhisms:

$$\mathrm{Hom}_{\mathcal{D}}(\mathrm{Lan}_{よ}\phi(P), d) \cong \mathrm{Hom}_{\mathcal{D}}(\int^{\mathcal{C}} \mathrm{Hom}_{\mathbf{C}}(よ(c), P) \cdot \phi(c), d) \cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathcal{D}}(\mathrm{Hom}_{\mathbf{C}}(よ(c), P) \cdot \phi c, d))$$

$$\cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}(\mathrm{Hom}_{\mathbf{C}}(よ(c), P), \mathrm{Hom}_{\mathcal{D}}(\phi c, d)) \cong \int_{\mathcal{C}} \mathrm{Hom}_{\mathrm{Set}}(Pc, \mathrm{Hom}_{\mathcal{D}}(\phi c, d)) \cong \mathrm{Nat}(P, \mathrm{Hom}(\phi(-), d)).$$

This concludes the proof □

We denote this right adjoint mapping $d$ to the presheaf $c \mapsto \mathrm{Hom}(\phi(c), d)$ by $N_\phi$ and call it the $\mathcal{D}$-coherent nerve. We also call $R_\phi$ the $\mathcal{D}$-realization functor.

In order to relate the above result to known material, the reader should replace $\mathcal{C}$ with the category $\Delta$ of finite ordered sets with order preserving maps (or rather a small category equivalent to this one), $\mathcal{D}$ the category of topological spaces and $\phi : \Delta \to \mathrm{Top}$ to be the natural map which on objects sends an $n$-element set to the standard $n$-simplex.
Now one can observe that the realization part of a "nerve-realization paradigm" admits a nicer description if we add a little bit of language.

**Definition 3.1.10.** (Section 4.1. in [24]) Suppose $\mathcal{V}$ is a Bénabou cosmos, $\mathcal{M}$ a symmetric monoidal $\mathcal{V}$-category and $\mathcal{D}$ a small category. Let $F : \mathcal{D} \to \mathcal{M}$ and $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ be functors. Then, in so far as the following coend exists, we define the following "tensor products of functors"

$$G \otimes_{\mathcal{D}} F = \int^{d \in \mathcal{D}} G(d) \cdot F(d).$$

This is an object in $\mathcal{M}$.

*Example* 3.1.11. This allows us to rewrite $R_\phi$ as $\mathrm{Hom}(よ(-), -) \otimes_{\mathcal{C}} \phi$. Specializing to the case of simplicial objects, i.e. in the case where $\mathcal{C} = \Delta$, we see that specifying a cosimplicial object $\Delta^\bullet : \Delta \to \mathcal{V}$ allows us to define a realization functor which sends a simplicial object $P$ to $\mathrm{Hom}(よ(-), P) \otimes_\Delta \Delta^\bullet$, which by the Yoneda lemma can also be rewritten as $P(-) \otimes_\Delta \Delta^\bullet$. This justifies writing realization of simplicial objects simply as a functor $- \otimes_\Delta \Delta^\bullet : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}$.

We justify the terminology of "tensor product" by the following example, which shows that it is indeed a generalization of the classical tensor product.

*Example* 3.1.12. (Example 3.6. in [19]) Any ring $R$ can be realized as a one object category enriched in abelian groups, denote the corresponding category by $BR$. Then a left $R$-module can be seen as a functor $M : BR \to \mathrm{Ab}$ and a right $R$-module as a functor $N : BR^{\mathrm{op}} \to \mathrm{Ab}$. We will in fact not need the fact that $BR$ is Ab enriched, and as we only discussed the unenriched version of the theory, we will forget the enrichment of $BR$.
The category of abelian groups is a symmetric monoidal Ab category, so the functor tensor product $M \otimes_{BR} N$ makes sense. It isn't hard to verify that this returns the classical tensor product of $R$-modules

All of the above discussion serves to see why a tensor product of functors can be thought of as both a generalization of geometric realization and of the ordinary tensor product, in particular this should give sufficient intuition for the following definition from Ching's paper.

**Definition 3.1.13.** (Definition 4.4 in [6]) Let $P$ be a reduced operad in a symmetric monoidal $\mathcal{T}$-category, where $\mathcal{T}$ is the category of topological spaces, then the bar construction $B(P)$ is the following symmetric sequence

$$B(P)(A) = \overline{w}(-) \otimes_{T(A)} P_A(-).$$

For the exact definition we let the reader consult Ching's paper. The intuition is based on what we have said so far and the connection between operads and trees. The object $A$ is a finite set, seen as an object in the core of the category of finite sets. The category $T(A)$ is a certain category of trees whose leaves are labeled by $A$. The functor $\overline{w}(-)$ is a natural geometric realization of the trees being considered and the functor $P_A(-)$ is a natural $T(A)$-shaped diagram built using $P$ and $A$, and so is the analogue of a simplicial set in "ordinary geometric realization". The $B$ stands for bar construction, which we discuss in the next subsection.

## 3.2 The bar construction

The bar construction is a relatively general way to "fatten" an object, which is proven to be useful in homotopy theory. We compare the definition given in Riehl's book [24] and in the paper [27] by Ruoqi Zhang, in particular giving the connection with the simplicial bar construction of an operad. Along with proposition 4.13. in Ching's paper, this relates general bar constructions with the bar construction on which Ching constructs a cooperad structure. All of this material can of course be dualized, we will not explicitly do this.

We immediately state the enriched bar construction, returning definition 4.2.1 in [24] when we consider Set-enriched categories.

**Definition 3.2.1.** (Definition 9.1.1. in [24]) Let $(\mathcal{V}, \wedge, I)$ be a Bénabou cosmos, $\mathcal{D}$ a $\mathcal{V}$-enriched category, where we denote the internal Hom by $\mathrm{Map}_{\mathcal{D}}$ and $\mathcal{M}$ a symmetric monoidal $\mathcal{V}$-category. Let $F : \mathcal{D} \to \mathcal{M}$ and $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$ be $\mathcal{V}$-functors. Then we define $B_\bullet(G, \mathcal{D}, F)$ the *enriched simplicial bar construction* to be the simplicial object in $\mathcal{M}$ whose $n$-simplices are given by

$$\bigsqcup_{D_0,\ldots,D_n \in \mathcal{D}} (GD_n \wedge \mathrm{Map}_{\mathcal{D}}(D_{n-1}, D_n) \wedge \mathrm{Map}_{\mathcal{D}}(D_{n-2}, D_{n-1}) \wedge \ldots \wedge \mathrm{Map}_{\mathcal{D}}(D_0, D_1)) \cdot FD_0.$$

The simplicial structure is given in a familiar way, where the degeneracy maps duplicate $D_i$ using the map $I \to \mathrm{Map}_{\mathcal{D}}(D_i, D_i)$, the inner face maps are given using composition and the outer face maps by evaluation maps. We let the interested reader consult [24] for details.

**Definition 3.2.2.** (Definition 9.1.5. in [24]) In the presence of a cosimplicial object $\Delta^\bullet : \Delta \to \mathcal{V}$, with the same notation as in the previous definiton, we define the *enriched bar construction* by

$$B(G, \mathcal{D}, F) = B_\bullet(G, \mathcal{D}, F) \otimes_\Delta \Delta^\bullet.$$

In spite of sharing the same name, the simplicial bar construction of definition 4.11 in Ching's paper bears no obvious resemblance to the above definition. More generally, while reading about the bar construction online, one stumbles regularly on "bar constructions" which resemble definition 4.11 in Ching's paper more than they do the above. Here is a definition of this alternative bar construction in relatively broad generality.

**Definition 3.2.3.** (Construction 5 in [27]) Let $(\mathcal{V}, \otimes, I)$ be a not necessarily symmetric monoidal category, let $G$ be a monoid in this category, $M$ a right $G$-object and $N$ a left $G$-object. The *(monoidal) simplicial bar construction* $B_\bullet(M, G, N)$ for the triple $(M, G.N)$ is the simplicial $\mathcal{V}$-object whose $q$-simplices is the object $M \otimes G^{\otimes q} \otimes N$. The interior face maps are given by internal multiplication, the first (resp. last) face maps by the $G$ action on $M$ (resp. on $N$) and the degeneracy maps are given by inserting a copy of $G$ using the unit map $I \to G$.
Just as for the simplicial bar construction defined above, in the presence of a cosimplicial object $\Delta^\circ : \Delta \to \mathcal{V}$, we can define the (monoidal) bar construction $B(M, G, N) = B_\bullet(M, G, N) \otimes_\Delta \Delta^\bullet$.

A special case of the above construction which is particularly common is when $(\mathcal{V}, \otimes, I)$ is $(\mathrm{Fun}(\mathcal{C}, \mathcal{C}), \circ, \mathrm{Id}_{\mathcal{C}})$. In this case, the term "monadic bar construction" is sometimes used. In this case, we may want the left module to be a $T$-algebra, i.e. an object $X \in \mathcal{V}$ with a natural transformation $\eta : TX \to X$. This can be recovered as a special case of the above by letting $N$ be the functor $X \otimes -$ and taking as the structure map the induced map $TX \otimes - \xrightarrow{\eta \otimes -} X \otimes -$.

The above constructions bears a passing resemblance to the bar construction discussed in [24], though it might not be immediately clear in what sense one generalizes the other.
The simplicial monoidal bar construction $B_\bullet(M, G, N)$ can be described via the simplicial enriched bar construction of [24] in the case that $\mathcal{V}$ is a Bénabou cosmos. In this case, we can view $G$ as a one object $\mathcal{V}$-enriched category, which we denote by $BG$ and $M$ can be viewed as a functor $BG \to \mathcal{V}$ and similarly $N$ can be viewed as a functor $BG^{\mathrm{op}} \to \mathcal{V}$.
In reality one might be able to lighten the assumptions on $\mathcal{V}$. The main assumption which cannot be

removed in an obvious manner is the fact that $\mathcal{V}$ needs to be symmetric. So that a priori the non-symmetric case belongs properly to the monoidal bar construction. On the other hand, it is obvious that if $\mathcal{D}$ is a single object category, then the bar construction of [24] corresponds to a monoidal bar construction, however the bar construction over more complicated $\mathcal{D}$ belongs properly to [24].

The simplicial operadic bar construction, for example as stated in definition 4.11 of Ching's paper is now best understood in the context of the monoidal bar construction, but because symmetric sequences form a symmetric monoidal category, it can in fact be reformulated in the language of [24]. Ching's paper also uses a bar construction for reduced operads of chain complexes, whose comparison with the above general setting is pointed out with remark 9.28 and proposition 9.29 in Ching's paper, which redirect the reader to section 4.4 in [8] and theorem 4.1.8.

We want to investigate the bar construction for operads of chain complexes over a field in more detail, with the goal of studying Koszul duality as our guiding light. We will in particular use this motivation to study what happens when we try and fiddle the bar construction into being a duality functor. Note that this is just a heuristic and serves as nothing more than vague motivation. In order to pursue this we follow [18], trying to motivate the construction via the above general setting, which we are able to do thanks to section 2.2. of [27]. The connection between the bar construction and Koszul duality is an important topic, starting in the classical formulation of Koszul duality in chapter 2 and 3 of [18], all the way to modern incarnations such as in section 5.2 of [21] or section 4 of [4]. For the sake of expositional clarity, instead of detailing the dg-operadic bar construction, we will detail the simpler case of dg-algebras in the following subsubsection §3.2.1.

### 3.2.1 Bar construction for dga

Let $A$ be a $k$-algebra. In order to have a natural bar construction, we need a right $A$-module $M$ and a left $A$-module $N$. In so far as we want to study $A$, we would like $M$ and $N$ not to affect the bar construction too much. As we are tensoring over $k$, we can achieve this if we could take $M = N = k$. Therefore, we assume further that $A$ is an augmented $k$-algebra.
The bar construction $B_\bullet(k, A, k) = k \otimes_k A^{\otimes q_\bullet} \otimes_k k$ is a simplicial $k$-vector space, and in general we cannot do much better. We could find or define some standard simplices, in order to get access to geometric realization. However because we are working in an Abelian category, we have another natural way to work with $B_\bullet(k, A, k)$, which is with the following theorem, called the Dold-Kan theorem.

**Theorem 3.2.4.** *(Theorem 8.4.1. in [26] and theorem 10 in [27]) For any abelian category $\mathcal{A}$ there is an equivalence of category*

$$N : \mathcal{A}^{\Delta^{\mathrm{op}}} \to \mathrm{Ch}_{\geq 0}(\mathcal{A}).$$

*This correspondence is given by sending $A_\bullet$ to the chain complex which in degree $n$ is $\bigcap_{i=0}^{n-1} \ker(d_i : A_n \to A_{n-1})$ and with differential $d_n$. Under this correspondence homotopy groups correspond to homology groups.*
*Furthermore, the complex $NA_*$ is equivalent to two other chain complexes which can be useful. First is the chain complex $A_*$ which in degree $n$ is $A_n$ and with differential $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$. Second is the quotient of $A_*$ by $DA_*$ which in degree $n$ is generated by the degenerated simplicies, i.e. $DA_* = \sum_{i=0}^{n-1} \mathrm{Im}(s_i)$. We call this quotient $CA_*$.*

The chain complex we choose to work with is $CB(k, A, k)_*$ as it has a tendency to be the most manageable in terms of size. One can check that in degree $n$ this is given by $k \otimes_k \bar{A}^{\otimes n} \otimes_k k$ where $\bar{A}$ is the augmentation ideal of $A$. Comparing definitions, we see that this can be identified with the cofree dg-coalgebra on the augmentation ideal of $A$ in degree 1, which is the definition of the bar construction given in [18]. The appearance of a coalgebra structure in the bar construction isn't suprising in cases where the free comonoid is well understood, as the fattening up done by the bar construction is philosophically similar to the "sum of possible decomposition of an element in the tensor algebra" which the cofree coalgebra captures.

With the goal of constructing a duality out of the bar construction, it is frustrating that we start with a $k$-algebra and end up with a dg-algebra. The above line of reasoning can be done for an algebra in any abelian category due to the generality of the Dold-Kan theorem, and so a naive fix to this problem would be to replace $k-\mathrm{Vect}$ with $\mathrm{Ch}(k-\mathrm{Vect})$, but then our construction would send a dg-algbera to an algebra in $\mathrm{Ch}(\mathrm{Ch}(k-\mathrm{Vect}))$. At this point one might expect the natural fix to be to work with infinite complexes, i.e. complexes of complexes of complexes of ... of vector spaces. But in fact we can stop at $\mathrm{Ch}(\mathrm{Ch}(k-\mathrm{Vect}))$, because we have a natural way to associate a complex to the double complex, given by taking the total complex.
We see that this construction takes a dg-algebra $(A_*, d)$ to a complex $(BA, d)$ which in degree $n$ is generated by elements $a_1 \otimes ... \otimes a_q, a_i \in \bar{A}$ such that $\sum_{i=1}^n |a_i| = n$ where $|a_i|$ is the degree of $a_i$ as an element of $A_*$. The differential is the sum of the differential coming from Dold Kan and the differential coming from $A_*$. For details on these differentials, we let the reader consult section 2.2 of [18].

The next key example for bar constructions is for dg-operads, which appear as a key technical component in the identification of the operad structure on the derivatives of the identity in [6]. As stated before detailing the bar construction for dg-algebras, we won't detail the operadic bar construction. We will use the definition of the bar construction for operads which appears in [18] in the next subsection, as this will be needed to discuss Koszul duality. However, we weren't able to prove the equivalence with the bar constructions we discussed in this subsection in a satisfactorily simple or conceptual manner. Therefore we opted not to include it in our general discussion, relying on the reader's willingness to verify for themselves that it fits within the framework of this subsection. We encourage the interested reader to consult [8], which treats everything that is needed in [6].

### 3.3 Koszul duality

Focusing our attention back down to an ordinary (i.e. not a dg-algebra) graded $k$-algebra $A$, we can apply the work we followed through in subsubsection §3.2.1 to the dg-algebra obtained by viewing $A$ as a dg-algebra concentrated in degree 0. Doing this we obtain a dg-algebra $(BA, d)$, and because of the absence of an internal differential, $d$ will be quite simple. In fact $d$ is given by

$$d(a_1 \otimes ... \otimes a_n) = \sum_{i=1}^{n} a_1 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n.$$

We have several competing notions of grading, notably the *homological* grading coming from the simplicial bar construction where $a_1 \otimes ... \otimes a_n$ has degree $n$ and the *weight* grading coming from the grading on $A$ where $a_1 \otimes ... \otimes a_n$ has degree $|a_1| + ... + |a_n|$ where $|a_i|$ is the degree of $a_i$ as an element of $A$. We see that our differential doesn't change the weight grading, so one might be tempted to study it with respect to the homological grading. However, with this grading an element in $A$ viewed in $BA$ is of degree 1, but if $BA$ has any chance of being a duality functor, we would rather have its homological data concentrated in degree 0.
Luckily, there is another grading, which incorporates both of the above gradings, called the syzygy grading, where $a_1 \otimes ... \otimes a_n$ has grading $|a_1| + ... + |a_n| - n$. With respect to this grading our differential clearly increases degree by 1. We denote the part of $BA$ which lives in syzygy degree $n$ by $B^n A$. Because we started with an algebra concentrated degree 0, one might hope that all the homological inforamation of this chain complex is contained in degree 0. This is made precise by the following definition.

**Definition 3.3.1.** (Section 3.4.7 in [18]) We call a graded algebra $A$ Koszul if the cochain complex $(B^\bullet A, d)$ has cohomology concentrated in degree 0. In this case we use the notation $A^{¡} = H^0(B^\bullet A, d)$ and call this the Koszul dual coalgebra. The Koszul dual algebra of $A$ is the $k$-linear dual of $A^{¡}$ and is denoted by $A^!$.

It turns out that this abstract characterization implies that the algebra $A$ is quadratic (see exercise 3.8.1. in [18]), i.e. of the form $T(V)/R$ where $R \subset V \otimes V$. Thanks to this, we may write a Koszul algebra as $A(V, R)$ as a short hand for $T(V)/R$, and with this notation the Koszul dual algebra and coalgebra admit a more explicit description

$$A(V, R)^{¡} = C(sV, s^2 R), \quad A(V, R)^! = A(V^*, R^\perp).$$

The notation $C(sV, s^2R)$ denotes a natural quadratic coalgebra associated to a vector space $V$ and "degree 2 relations" $R$. Details are available in section 3 of [18], in particular we encourage the interested reader to consult section 3.1 to 3.3. In particular, anyone following their curiosity up to section 3.2.3. will feel justified in calling this notion a duality.

At this point we have two heuristic perspectives for Koszul duality, the first being the one announced in the previous section as what happens when one tries really hard to make the bar construction a duality functor and the other as taking an algebraic object with generators and relations and replacing the relations by orthogonal ones. We now move on to discussing Koszul duality for operads.

The Koszul duality of operads draws heavy inspiration from the Koszul duality of algebras. In particular, we work directly with quadratic operads, instead of defining a a priori more general theory, before realizing that the conditions we require impose being quadratic.

**Definition 3.3.2.** (Section 7.1 in [18]) We call a pair $(E, R)$ an *operadic quadratic data* when $E$ is a symmetric sequence of $k$-modules with $R$ a sub-symmetric sequence of the free operad $T(E)$ on $E$ which lives entirely in degree 2. This can be used to define an operad $P(E, R) = T(E)/R$.
We call an operad $P$ *quadratic* if it is isomorphic to an operad of the form $P(E, R)$.

Just as in the algebra case there is an analgous construction of quadratic cooperads, which with $E$ and $R$ just as in the above definition, we denote by $C(E, R)$. In perfect analogy with the algebra case, for a quadratic operad $P \cong P(E,R)$ we call $P^{\text{i}} := C(sE, s^2R)$ its Koszul dual cooperad. Defining the Koszul dual operad of an operad takes a bit more work. For $P \cong P(E,R)$, it is given in symbols by

$$P^! = (\mathrm{End}_{s\mathbb{K}} \otimes_H P^{\text{i}})^*.$$

To make sense of the above notation we refer the reader to section 5.3.2. and 7.2.2. of [18]. The fact that this in fact deserves to be called a duality functor is given by the following statement.

**Theorem 3.3.3.** *(7.2.5. in [18]) For any quadratic operad $P = P(E,R)$ such that each $E_n$ is finite dimensional we have*

$$(P^!)^! \cong P.$$

In analogy with the algebra case, we would like a criterion to tell whether an operad is Koszul, in particular relating it to the appropriate bar construction. A priori, there are several different definitions of the bar construction for an operad, the unification of which can be unclear. The bar construction we consider is given by mapping an augmented operad $P$ to the cofree algebra on the suspension of the augmentation ideal of $P$. The operad $BP$ can be equipped with a syzygy grading in much the same way as the algebra case, and the natural differential raises this degree by 1. And so similarly to the algebra case, we have the following result.

**Proposition 3.3.4.** *(Proposition 7.3.2. in [18]) Let $(E,R)$ be an operadic quadratic data, and $P$ the associated quadratic operad. There is a natural inclusion of cooperads $i : P^i \to BP$, which induces an isomorphism of operads*

$$P^i \to H^0(BP, d).$$

This idea is then pursued in section 5.2 of [8], which gives the following definition of the Koszul dual of an augmented (not necesairly quadratic) dg-operad.

**Definition 3.3.5.** (5.2.3. in [8]) Consider a reduced dg-operad $P_*$, where the subscript is meant to indicate the dg-degree. In the bar construction $B_\bullet P_*$ , the symbol $\bullet$ is a placeholder for the tree degree. Then the *Koszul dual* of $P_*$ is given as

$$K(P)_s = H_s(B_*(P)_s, d),$$

where $d$ is the differential coming from the bar construction.

If we place the additional restrictions on $P$ that it has trivial differential and is concentrated in dg-degree 0, the above definition is equivalent to the one which is used in Ching's paper. Section 5.2.5 of [8] gives the connection between the more general theory of Koszul duality and the one in [18] we just briefly discussed. In particular, for quadratic operads, we may use the explicit description of $P^!$. Furthermore, lemma 5.2.10. of [8] shows that there is a posteriori no loss of generality by restricting to the case of quadratic operads. This same lemma would be another way to say that in the case of Koszul operads, the term duality is justified.
However for a truly complete understanding, one needs the bar construction/homological perspective on Koszul duality, as it is only with this perspective that the following result from Ching's paper can be proven.

**Theorem 3.3.6.** *(Proposition 9.48 in [6]) Let $P$ be a reduced operad in spaces or spectra, such that each object $P(A)$ is cofibrant and all of the homology groups with coefficients in $k$ $H_*(P(A))$ and $H_*(B(P)(A))$ are flat $k$-modules. Then if we further have that $H_*(P)$ is Koszul, then*

$$H_*(B(P)) \cong K(H_*(P)).$$

There is much more, which we unfortunately will not have time to discuss. In particular, as we have already somewhat alluded to, it isn't immediately obvious how the bar construction of Loday and Valette [18] based on the cofree cooperad relates to the tree based bar construction of Ching and Freese ([6] and [8]). One substantial difference is that [18] works with a cohomological grading (the syzygy grading) and [6] and [8] work with a homological "tree grading". The connection between these can only be discussed once we understand the relationship between the two bar constructions. In order to achieve this, the most natural path is to connect the bar construction of [18] with the monoidal bar construction, as in theorem 4.1.8 of his paper [8], where Freese shows the connection between his bar construction and the simplicial bar construction. Another approach, which could allow for more abstract arguments, would be by using universal properties. Although these already appeared in [18] via twisting morphisms, they aren't emphasized. With more modern perspectives, this becomes central, for example in section 5.2 of [21].

### 3.4 Derivatives of the identity

In this subsection we aim to have at least a vague understanding of the computation of the derivatives of the identity $\mathrm{Id} : \mathcal{S} \to \mathcal{S}$ as done by Johnson in [16]. The model of the derivatives of the identity used in Ching's paper to understand the operad structure is a slightly upgraded version which is described by Arone and Mahowald in [1]. This description is used by Ching to obtain his corollary 8.8, which gives the following concise description of the derivatives of the identity

$$\partial_* \mathrm{Id} = \Omega(\mathbb{D}\underline{S^0}).$$

Where $\Omega$ is the cobar construction on the cooperad $\mathbb{D}\underline{S^0}$, which is the Spanier-Whitehead dual of the operad of based spaces which is constant equal to $S^0$ in each arity. This result, coupled with the dual of theorem 3.3.6 and a simple computation is what justifies the claim that the derivatives of the identity are an analogue in the category of spectra to the Lie operad.
The reason we will discuss the computation in [16] instead of the one in [1] is because a thorough understanding of either of these papers is not achievable within the span of this project, and so instead we rely on notes by Ben Knudsen [3] which give an overview of the construction in [16].

The first step in computing the derivatives of the identity is the following result. The point of this result is that $\mathrm{cr}_n(D_n(F))$ is the key ingredient in the definition of $\partial_n F$, but is rather unwieldy, whereas there is a chance at direct inspection for $P_{1,\ldots,1}\,\mathrm{cr}_n(F)$.

**Proposition 3.4.1.** *(Remark 6.1.3.23. in [21]) Let $\mathcal{C}$ be an $\infty$-category which admits colimits and a final object, let $\mathcal{D}$ be a pointed differentiable category and let $F : \mathcal{C} \to \mathcal{D}$ be a reduced functor. Then we have*

$$P_{1,\ldots,1}\,\mathrm{cr}_n(F) \simeq \mathrm{cr}_n(P_n(F)),$$

*which implies*

$$\mathrm{cr}_n(D_n(F)) \simeq P_{1,\ldots,1}(\mathrm{cr}_n(F)).$$

*Proof.* This proof is essentially an exercise in recollection of results from the first chapter of this project. Recall that $P_n$ is left exact by lemma 1.2.7, that the $n$th cross effect of $F$ is given by $\mathrm{Red}(F \circ q)$ where $q : \mathcal{C}^n \to \mathcal{C}$ is the coproduct functor and Red is defined via a limit in definition 2.1.7. It is clear that $P_n$ commutes with precomposition by $q$ by lemma 1.2.9, so that we may deduce $\mathrm{cr}_n(P_n F) = \mathrm{Red}(P_n(F) \circ q) \simeq \mathrm{Red}(P_n(F \circ q)) \simeq P_n(\mathrm{Red}(F \circ q)) = P_n(\mathrm{cr}_n F)$ which implies the first part of the desired result by the lemma below.
Now, one can observe that $\mathrm{cr}_n$ is left exact. To see this, first notice that this can be verified object wise. Inspecting the definition, the cross effect of $F$ at an $n$-tuple is given by a limit over a diagram whose objects are all $F$ evaluated at some coproduct of elements of the $n$-tuple. So the result follows because limits commute with limits and with evaluation at an object. This implies that $\mathrm{cr}_n(D_n F) = \mathrm{cr}_n(\mathrm{fib}(P_n F \to P_{n-1} F)) \simeq \mathrm{fib}(\mathrm{cr}_n(P_n F) \to \mathrm{cr}_n(P_{n-1} F))$. Which by the above paragraph proves the desired result assuming that $\mathrm{cr}_n(P_{n-1} F) = \underline{*}$, i.e. the $n$th crosseffect of an $n-1$ excisive functor is constant equal to the terminal object of $\mathcal{D}$. This follows from a direct application of proposition 2.1.13, which shows that $\mathrm{cr}_n(P_{n-1} F)$ is 0-excisive, i.e. constant, which indeed gives the desired result because $F$ is reduced ∎

Now in order to understand $P_{1,\ldots,1}(\mathrm{cr}_n(F))$, we wish to reduce the somewhat easier single variable Goodwillie calculus. We do this with the corrolary which follows the following result.

**Lemma 3.4.2.** *(6.1.3.13. in [21]) Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be $\infty$-categories with finite colimits and final objects, let $\mathcal{D}$ be an $\infty$-category with finite limits and let $F : \mathcal{C}_1 \times \ldots \times \mathcal{C}_m \to \mathcal{D}$ be a functor reduced in each variable. Let $\mathcal{C} = \mathcal{C}_1 \times \ldots \times \mathcal{C}_m$ and let $F'$ be the functor $F$ seen as a single variable functor. If $F'$ is $m$-excisive, then $F$ is $(1, \ldots, 1)$-excisive*

*Proof.* Without loss of generality we can show that $F$ is 1-excisive in its first variable with all other variables constant. That is we want to show that given $X_i \in \mathcal{C}_i, \forall i \in \{2, ..., n\}$ and a pushout square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array},$$

then the following square is a pullback

$$\begin{array}{ccc} F(Y, X_2, ..., X_n) & \longleftarrow & F(Z, X_2, ..., X_n) \\ \uparrow & & \uparrow \\ F(Y', X_2, ..., X_n) & \longleftarrow & F(Z', X_2, ..., X_n) \end{array}.$$

To use the $m$-excisiveness of $F$, we want to build an $m$-cube out of the square

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}.$$

In order to achieve this, for $i \in \{2, ..., m\}$ choose maps $X_i \to *_i$ where $*_i \in \mathcal{C}_i$ is a final object, which we view as maps $\tau_i : \Delta^1 \to \mathcal{C}_i$ and view the above pushout as a map $\sigma : \Delta^1 \times \Delta^1 \to \mathcal{C}^1$. Taking the product $\sigma \times \tau_2 \times ... \times \tau_n$ we obtain a strongly coCartesian $m$-cube $U : \mathrm{N}(\mathcal{P}([m])) \to \mathcal{C}$ (the fact that this is indeed strongly coCartesian follows from lemma 1.1.4). Now by assumption, $F'(U)$ is a Cartesian cube in $\mathcal{D}$. Because $F$ is reduced in each variable, we have that $F(U)(T)$ is a final object of $\mathcal{D}$ unless $T \subset \{0, 1\}$. So the Cartesian square $F(U)$ is a right Kan extension (see definition A.0.4) of its restriction to $\mathrm{N}(\mathcal{P}([1]))$. This in particular implies that $F(U)|_{\mathrm{N}(\mathcal{P}([1]))}$ is a pullback, which is what we wanted to show. □

**Lemma 3.4.3.** *(Corollary 6.1.3.14. in [21]) Let $\{\mathcal{C}_i\}_{i=1}^m$ be $\infty$-categories which admit finite colimits and a final object, let $\mathcal{D}$ be a differentiable category and let $F : \prod_{i=1}^m \mathcal{C}_i \to \mathcal{D}$ be a functor which is reduced in each variable. Let $\mathcal{C} = \prod_{i=1}^m \mathcal{C}_i$ and let $F' : \mathcal{C} \to \mathcal{D}$ be the functor $F$ viewed as a single variable functor. Then there is a cannonical equivalence $P_m F' \cong P_{1,...,1} F$.*

*Proof.* We know that the $(1, ..., 1)$-excivieness of $F$ implies that $F'$ is $m$-excisive by proposition 2.1.4. By universal property of $P_m(F')$ this means that the natural map $F' = F \to P_{(1,..,1)}F$ factors through $P_m F'$, so that i in particular we get a map $\alpha : P_m F' \to P_{(1,...,1)}F$.
If we could show that $P_m F'$ is reduced in each variable, we could deduce by the above result that $P_m F'$ is $(1, .., 1)$-excisive, which would imply that the natural map $F = F' \to P_m F'$ factors through $P_{(1,...,1)}F$, in particular yielding a map $\beta : P_{(1,...,1)}F \to P_m F'$. The usual argument for maps obtained from universal properties in this way imply that in this case $\alpha$ and $\beta$ are mutually inverse.
Thus it suffices to show that $P_m F'$ is reduced in each variable. Let $\mathcal{E}_i \subset \mathcal{C}$ be the full subcategory spanned by those $m$-tuples whose $i$th coordinate is a final object. The inclusion $\mathcal{E}_i \to \mathcal{C}$ preserves pushouts, so that by lemma 1.2.9 we get $P_m(F')|_{\mathcal{E}_i} \cong P_m(F'|_{\mathcal{E}_i})$. Now because $F'$ is reduced in each variable we have that $F'|_{\mathcal{E}_i}$ is constant equal to a final object of $\mathcal{D}$, which implies that the same holds for $P_m(F'|_{\mathcal{E}_i})$. Thus $P_m F'$ maps any $m$-tuple with $i$th coordinate a final object to a final object, so that $P_m F'$ is reduced in its $i$th variable. Because $i$ was arbitrary this concludes the proof. □

And so by inspecting the definition of the derivative, we immediately derive the following key result

**Corollary 3.4.4.** *(Proposition 8 in [3]) Let $F : \mathcal{S}_* \to \mathcal{S}_*$ be a reduced functor from based spaces to based spaces. Then there are natural $\Sigma_n$-equivariant equivalences*

$$\Omega^\infty \partial_n F(\mathbb{S}, ..., \mathbb{S}) \simeq \varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} \operatorname{cr}_n(F)(S^{k_1}, ..., S^{k_n}).$$

*The action of $\Sigma_n$ on the left side is induced from the permutation of argument action on $\partial_n F : \mathrm{Sp}(\mathcal{C})^n \to \mathrm{Sp}(\mathcal{D})$ and the action on the right comes from the natural action $\Sigma_n \curvearrowright \mathbb{N}^n$.*

Recalling that $\partial_n F$ is fully determined by where it sends $(\mathbb{S}, ..., \mathbb{S})$, this greatly simplifies our study of the derivatives of the identity. In particular it suffices to understand

$$P_{1,...,1}\,\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(S^0, ..., S^0) \cong \varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n}\,\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(S^{k_1}, ..., S^{k_n}).$$

We start by studying $\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})$, which recalling (and slightly unwinding) the definition we see that, evaluated at $(X_0, ..., X_n)$, it yields the fiber of the natural map

$$X(\emptyset) \to \varprojlim_{\emptyset \neq S \subset [n]} X(S),$$

where $X : \mathcal{P}([n]) \to \mathcal{S}_*$ maps a subset $S$ to $\bigvee_{i \notin S} X_i$. In general, for an $n$-cube $X$, we call the above fiber the *total fiber* of $X$ and denote it by $\mathrm{tfib}(X)$. We want to understand the total fiber better, in particular and explicit model, and in order to do that we introduce some notation.

**Definition 3.4.5.** (Notation 9 in [3]) Let $I$ be a finite set, for $S \subset I$ we write

$$[0,1]^S = \{t \in [0,1]^I \mid i \notin S \Rightarrow t_i = 0\}$$

and

$$\partial_1[0,1]^S = \{t \in [0,1]^S \mid \exists i \in S \text{ such that } t_i = 1\}.$$

It is clear that both of these construction give us $I$-cubes $[0,1]^\bullet : \mathcal{P}(I) \to \mathcal{S}_*$ and $\partial_1[0,1]^\bullet : \mathcal{P}(I) \to \mathcal{S}_*$.

This allows us to state the following result.

**Proposition 3.4.6.** *(Lemma 10 in [3] and 5.5.8 in [23]) Let $X : \mathcal{P}(I) \to \mathcal{S}_*$ be an $I$-cube, then we have the following (strict) pullback*

$$\begin{array}{ccc} \mathrm{tfib}(X) & \longrightarrow & \mathrm{Nat}([0,1]^\bullet, X) \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & \mathrm{Nat}(\partial_1[0,1]^\bullet, X) \end{array}.$$

*The right vertical map is the restriction map, and the bottom map is the inclusion as the natural transformation constant equal to the base point.*
*Further this pullback is also a homotopy pullback.*

*Proof.* Denote by $n$ the cardinality of $I$, and identify $I$ with $\{1, ..., n\}$. Notice that the case $n = 1$, this is just an explicit model for the fiber of the map $X(\emptyset) \to X(\{1\})$ (see for example definition 2.2.1. of [23]). This opens the way for a proof by induction, so hence forth assume the result holds for $(n-1)$ cubes and let $X$ be an $n$-cube of topological spaces. We can view $X$ as a morphism of $(n-1)$ cubes, from the face corresponding to subsets not containing $n$ to the face corresponding to the subsets containing $n$. We write this morphism as $\chi : X_a \to X_b$ to fix some notation. It now follows from comparison of universal property that

$$\mathrm{tfib}(X) \simeq \mathrm{fib}(\mathrm{tfib}(X_a)) \xrightarrow{\chi} \mathrm{tfib}(X_b)).$$

To clear up what we mean by this, we specify that the base point of the total fiber of an $(n-1)$-cube is given by the natural transformation which maps $[0,1]^S$ constantly to the base point of $Y(S)$. Now using the induction hypothesis and the fact that we have an explicit model for the homotopy fiber we get that the total fiber of $X$ can be described as the subspace of

$$\mathrm{tfib}(X_a) \times \mathrm{Map}_*([0,1], \mathrm{tfib}(X_b)),$$

consisting of those pairs $(\{f_S\}_{S\subset\{1,...,n-1\}}, \{g_{t,S\cup\{n\}}\}_{S\subset\{1,...,n-1\}})$ such that $g_{0,S\cup\{n\}} = \chi_S \circ f_S$. Equivalently, this is for each $S \subset \{1, ..., n\}$ a map $h_S : [0,1]^S \to X(S)$, where depending on whetehr $n \in S$ or not we have $h_S = g_{t,S\setminus\{n\}}$ or $h_S = f_S$. Furthermore, for any inclusion of subset $T \subset S$, we have that $h_S|_{[0,1]^T} = X(T \subset S) \circ h_T$. To see this, there are 3 cases: the first case is $n \in T \subset S$, so that the desired relation holds because it holds for the $g_S$; in the second case, $n \notin S$, so that the desired relation holds because it holds for $f_S$; and the third case is when $n \in S \setminus T$, by the previous two cases we only need to deal with the case $S = T \cup \{n\}$ in which case the desired relation follows from $g_{0,S\cup\{n\}} = \chi_S \circ f_S$. $\square$

Explicitly, this means the total fiber of $X$ can be identified with the collection of maps $\{f_S : [0,1]^S \to X(S)\}_{S\subset I}$ such that the following diagram commutes

$$\begin{array}{ccc} [0,1]^S & \xrightarrow{f_S} & X(S) \\ \downarrow & & \downarrow \\ [0,1]^T & \xrightarrow{f_T} & X(T) \end{array}$$

and such $f_S(t)$ is the base point of $X(S)$ whenever one of the $t_i$ equals 1. What this description allows us to do is to construct the map, which will in due time allow us to compute the derivatives of the identity. Indeed for a fixed index $i \in I$ we consider the map which sends an $I$-tuple $\{f_S : [0,1]^S \to X(S)\}_{S\subset I}$ to $f_{I\backslash\{i\}} : [0,1]^{|I|-1} \to X(I\backslash\{i\})$. Of course, we can take the product over all $i \in I$, which yields a map $\text{tfib}(X) \to \text{Map}_*([0,1]^{|I|(|I|-1)}, \prod_{i\in I} X(I\backslash\{i\}))$. Recalling that we are working in the pointed category, post composing by the quotient $\prod_{i\in I} X(I\backslash\{i\}) \to \bigwedge X(I\backslash\{i\})$ we get the *comparison map* $\text{tfib}(X) \to \text{Map}([0,1]^{|I|(|I|-1)}, \bigwedge_{i\in I} X(I\backslash\{i\}))$.
Of course we want to apply this construction to the $n$-cube defining the cross effect, which yields a map

$$\gamma : \text{cr}_n(\text{Id}_{\mathcal{S}_*})(X_1, ..., X_n) \to \text{Map}_*([0,1]^{n(n-1)}, \wedge_{i=1}^n X_i).$$

It is clear by construction that the $\gamma$ assemble into a natural transformation $\text{cr}_n(\text{Id}_{\mathcal{S}_*})(-) \to \text{Map}_*([0,1]^{n(n-1)}, \wedge_{i=1}^n(-))$ of functors $\mathcal{S}_*^n \to \mathcal{S}_*$. Recall that our goal is to find a more direct description of the derivatives of the identity, in order to increase the chances that $\gamma$ will give us an equivalence describing the derivatives of the identity, we would like to refine this map. What we mean by this, is that we would like to corestrict this map, which can be done if there is a subspace $W \subset [0,1]^{n(n-1)}$ such that every continuous map $[0,1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ in the image of $\gamma$ maps $W$ to the base point of $\bigwedge_{i=1}^n X_i$. Indeed, if this is the case we will be able to correstrict $\gamma$ to $\text{Map}_*([0,1]^{n(n-1)}/W, \bigwedge_{i=1}^n X_i)$.
To do this, for the sake of convenience, we will view $[0,1]^{n(n-1)}$ as the space of $n \times n$ matrices with coefficients in $[0,1]$ with 0 on the diagonal, so that for an element $t \in [0,1]^{n(n-1)}$ the notation $t_{ij}$ makes sense. In this notation, the $i$th row of the matrix corresponds to the copy of $[0,1]^{n-1}$ coming from $\{1, ..., n\}\backslash\{i\}$.
The second matter of convenience is the following definition, a pedagogical tool at the service of the next proposition.

**Definition 3.4.7.** We call $t \in [0,1]^{n(n-1)}$ "irrelevant" if every continuous map $[0,1]^{n(n-1)} \to \bigwedge_{i=1}^n X_i$ in the image of $\gamma : \text{cr}_n(\text{Id}_{\mathcal{S}_*})(X_1, ..., X_n) \to \text{Map}_*([0,1]^{n(n-1)}, \wedge_{i=1}^n X_i)$ maps $t$ to the base point of $\bigwedge_{i=1}^n X_i$.

We have the following (not necessarily exhaustive) description of irrelevant points

**Proposition 3.4.8.** *(Lemma 12 in [3]) The $t \in [0,1]^{n(n-1)}$ such that $t_{ij} = 1$ for some $i, j \in \{1, ..., n\}$ are irrelevant. Further the $t \in [0,1]^{n(n-1)}$ such that $t_{ik} = t_{jk}$ for all $k \in \{1, ..., n\}$ and some fixed and distinct $i, j \in \{1, ..., n\}$ are irrelevant.*

*Proof.* The first part of the claim is immediate from proposition 3.4.6. So we need only prove the second claim.
For the second part, denote the subspace of $t \in [0,1]^{n(n-1)}$ such that $t_{ik} = t_{jk}$ for all $k \in \{1, ..., n\}$ by $W_{ij}$. We take an element $\{f_S\}_{S\subset\{1,...,n\}} \in \text{cr}_n(\text{Id}_{\mathcal{S}_*})(X_1, ..., X_n)$. We need to show that for $t \in W_{ij}$ each $f_{I\backslash\{i\}}(t)$ is equal to the base point of $X_i$. For any $S \subset \{1, ..., n\}$ there is a map $[0,1]^{n(n-1)} \to [0,1]_S^{n(n-1)}$ given by discarding all the rows whose index isn't in $S$, we denote these maps by $\pi$, leaving the dependance on $S$ clear from context. We have the following clear, though somewhat index heavy

commutative diagram

$$\begin{tikzcd}
 & {[0,1]^{n(n-1)}} & \\
 & W_{ij} \arrow[u] \arrow[d, "\pi|_{W_{ij}}"] & \\
{[0,1]^{n(n-1)}_{\{i\}}} \arrow[d, "f_{\{1,...,n\}\backslash\{i\}}"'] & {[0,1]^{n(n-1)}_{\{i,j\}} \cap W_{ij}} \arrow[l] \arrow[r] \arrow[d, "f_{\{1,...,n\}\backslash\{i,j\}}"] & {[0,1]^{n(n-1)}_{\{j\}}} \arrow[d, "f_{\{1,...,n\}\backslash\{j\}}"] \\
X_i & X_i \vee X_j \arrow[l] \arrow[r] & X_j
\arrow[from=1-2, to=3-1, "\pi"'] \arrow[from=1-2, to=3-3, "\pi"]
\end{tikzcd}.$$

The commutativity of the triangles is a consequence of the construction of $\pi$ and the commutativity of the rectangles follows from the assumption that the vertical maps come from $\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(X_1, ..., X_n)$ which we view as a subspace of $\mathrm{Nat}([0,1]^\bullet, \bigvee_{i \notin \bullet} X_i)$. Now for a given $t \in W_{ij}$, it is mapped by $f_{\{1,...,n\}\backslash\{i,j\}}$ either to $X_i$ or to $X_j$ (or potentially both if $t$ is mapped to the based point). Without loss of generality, let us assume $t$ is mapped to $X_i$, then by commutativity of the diagram, we see that $f_{\{1,...,n\}\backslash\{j\}}(t)$ must be the base point of $X_j$. This is enough for $\{f_S\}_{S\subset\{1,...,n\}} \in \mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(X_1, ..., X_n)$ to map $t \in W_{ij}$ to the base point of the smash product. This shows the desired claim. $\square$

We denote the quotient of $[0,1]^{n(n-1)}$ by the subspace of the irrelevant $t$ described in the above proposition by $\Delta_n$ and the natural transformation $\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*}) \to \mathrm{Map}_*(\Delta_n, \bigwedge_{i=1}^n(-))$ by $\phi$. Notice that this natural transformation is $\Sigma_n$-equivariant, where the action on the domain an codomain are by permutation of variables. The claim we are going to work towards is that the map induced by $\phi$ on the following colimits

$$\varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} \mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(S^{k_1}, ..., S^{k_n}) \xrightarrow{\phi} \varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} \mathrm{Map}_*(\Delta_n, \bigwedge_{i=1}^n S^{k_i})$$

is an equivalence. For a functor $F : \mathcal{S}_*^n \to \mathcal{S}_*$ we call the colimit

$$\varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} F(S^{k_1}, ..., S^{k_n})$$

its multilinearization. With this vocabulary in hand, our goal has now become to show that the induced map by $\phi$ on multilinearizations is a weak equivalence.

**Lemma 3.4.9.** *(Lemma 15 in [3]) Let $F, G : \mathcal{S}_*^n \to \mathcal{S}_*$ be two functors and $\psi : F \to G$ be a natural transformation between them. If $\psi_{(X_1,...,X_n)}$ is $((n+1)k-c)$-connected whenever each $X_i$ is $k$-connected, then $\psi$ induces a weak equivalence after multilinearization.*

*Proof.* For the sake of compactness, we introduce the notation $F_l = \Omega^{nl}F(\Sigma^l X_1, ..., \Sigma^l X_n)$, $G_l = \Omega^{nl}G(\Sigma^l X_1, ..., \Sigma^l X_n)$ and $\psi_l : F_l \to G_l$ the map $\Omega^{nl}\psi_{(\Sigma^l X_1,...,\Sigma^l X_n)}$. The map we are interested in is the induced map on colimits $\psi_\infty : F_\infty \to G_\infty$, we want to show this map is an equivalence. The connectivity assumptions yield that $\psi_l$ is at least $((n+1)l - c - nl)$-connected, which tends to infinity when $l$ does. So that in particular, the induced map on $\pi_m$ is an isomorphism for large enough $l$. Now compactness of spheres implies that $\pi_m$ commutes with filtered colimits, and so because $\pi_m(\psi_l)$ is eventually an isomorphism we have that $\pi_m(\psi_\infty)$ is an isomorphism for all $m$. This prove the desired claim that $\psi_\infty$ is a weak equivalence. $\square$

The above result has the following corollary which is the result we will actually use to show that $\phi$ is a weak equivalence after multilinearization.

**Corollary 3.4.10.** *(Corollary 16 in [3]) Let $F, G : \mathcal{S}_*^n \to \mathcal{S}_*$ be two functors and $\psi : F \to G$ be a natural transformation between them. If $\Omega\psi_{(\Sigma X_1,...,\Sigma X_n)}$ is $((n+1)k - c)$-connected whenever each $X_i$ is $k$-connected, then $\psi$ induces a weak equivalence after multilinearization.*

*Proof.* By the previous result, the assumptions imply that the whiskering $\Omega * \psi * (\Sigma, ..., \Sigma))$ induces an equivalence after multilinearization. So the claim follows from the fact that $\psi$ and $\Omega * \psi * (\Sigma, ..., \Sigma))$ clearly have the same multilinearization. $\square$

The reason we will use the above result is that studying the connectivity of $\phi$ directly turns out to be quite tricky, whereas the new map we obtain by post composing by $\Omega$ and precomposing coordinate wise by $\Sigma$ is much more approachable. The reason for this (or at least part of the reason) is that the homotopy groups (which we need to understand at least enough to prove the connectivity assumptions) of a wedge can be quite hard to understand, whereas by the following computation

$$\begin{aligned}\Omega\,\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(\Sigma X_1, ..., \Sigma X_n) &\simeq \Omega\,\mathrm{tfib}(S \mapsto \bigvee_{i\notin S\subset\{1,...,n\}} \Sigma X_i) \\ &\simeq \Omega\,\mathrm{tfib}(S \mapsto \Sigma \bigvee_{i\notin S\subset\{1,...,n\}} X_i) \\ &\simeq \mathrm{tfib}(S \mapsto \Omega\Sigma \bigvee_{i\notin S\subset\{1,...,n\}} X_i) \\ &\simeq \mathrm{cr}_n(\Omega\Sigma)(X_1, ..., X_n)\end{aligned}$$

the homotopy groups we need to compute when we post/precompose appropriately can be attacked with the Hilton-Milnor theorem. For the sake of completeness, we recall it here.

**Theorem 3.4.11.** *(Theorem 17 in [3])(Theorem 5.9 in [17]) Let $\{X_i\}_{i=1}^n$ be a collection of connected pointed spaces. There is a canonical natural weak equivalence*

$$\prod_{w\in L_n}{}' \Omega\Sigma(w(X_1, ..., X_n)) \xrightarrow{\sim} \Omega\Sigma(X_1 \vee ... \vee X_n),$$

*where $L_n$ denotes the set of of "basic products in the free Lie algebra on $n$-generators $\langle x_1, ..., x_n\rangle$, the notation $\prod'$ denotes the "weak infinite product" and $w(X_1, ..., X_n)$ is obtained from $w$ by replacing each instance of $x_i$ by $X_i$ and the Lie bracket by the smash product.*

We won't detail the above theorem anymore than this as we won't need a perfect comprehension of its inner workings to apply it. The only details we will use is a count of a certain subset of basic words, which we will state when we need it in the proof of the following computation.

**Lemma 3.4.12.** *(Corollary 18 in [3]) Let $\{X_i\}_{i=1}^n$ be a collection of $k$-connected pointed spaces, then*

$$\pi_m(\mathrm{cr}_n(\Omega\Sigma)(X_1, ..., X_n)) \cong \pi_m(\bigwedge_{i=1}^n X_i)^{(n-1)!}$$

*for $0 \leq m \leq (n+1)(k+1) - 1$.*

*Proof.* We want to understand $\pi_m$ of $\mathrm{tfib}(S \mapsto \Omega\Sigma(\bigvee_{i\notin S\subset\{1,...,n\}} X_i))$, and so naturally, we apply the Hilton-Milnor theorem to instead study

$$\mathrm{tfib}(S \mapsto \prod_{L_{|\{1,...,n\}\setminus S}}{}' \Omega\Sigma(w(X_1, ..., X_n))).$$

Direct inspection via the definition of the total fiber of $X$ as $\mathrm{fib}(X(\emptyset) \to \varinjlim_{\emptyset\neq S\subset I} X(S))$ gives an equivalence

$$\mathrm{tfib}(S \mapsto \prod_{L_{|\{1,...,n\}\setminus S}}{}' \Omega\Sigma(w(X_1, ..., X_n))) \cong \prod_{w\in L_n^\circ} \Omega\Sigma(w(X_1, ..., X_n)),$$

Where $L_\circ^n$ consists of those basic product involving each $x_i$ at least once. Because $\pi_m$ commutes with finite products and filtered colimits, it commutes with the weak infinite product, so that we may reduce our study to $\pi_m(\Omega\Sigma(w(X_1, ..., X_n)))$ for $w \in L^n_\circ$. Observe that for the $w \in L^n_\circ$ containing a certain $x_i$ more than once, because smash product sums connectivity, we have that $\Omega\Sigma w(X_1, ..., X_n)$ is at least $((n+1)k-1)$-connected. In particular these don't contribute anything to $\pi_m$ in the range we are interested in. This yields an isomorphism

$$\pi_m(\mathrm{cr}_n(\Omega\Sigma)(X_1, ..., X_n)) \cong \prod_{L^\star_n}{}' \pi_m(\Omega\Sigma(w(X_1, ..., X_n))),$$

where $L^\star_n$ consists of the basic words where each $x_i$ appears exactly once. This implies that $w(X_1, ..., X_n) \cong \bigwedge_{i=1}^n X_i$ for all the $w$ we are considering. So the desired result follows from the fact that there are $(n-1)!$ basic products containing each $x_i$ exactly once. □

To show that $\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}$ is connected enough, it would be great if we could show that $\pi_m(\Omega\,\mathrm{Map}_*(\Delta_m, \bigwedge_{i=1}^n \Sigma X_i)) = \pi_m(\bigwedge_{i=1}^n X_i)^{(n-1)!}$, in the range $0 \leq m \leq (n+1)(k+1)-1$. Indeed, in that case, the homotopy groups of the domain and codomain of $\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}$ would be abstractly isomorphic, therefore, up to showing that the isomorphism is realized by $\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}$, we would have that this map is $((n+1)(k+1)-1)$ connected, which is enough to apply corollary 3.4.10. There is a very similar space that has the desired homotopy groups, indeed we have

$$\pi_m(\bigwedge_{i=1}^n X_i)^{(n-1)!} \cong \pi_{m+n}(\bigwedge_{i=1}^n \Sigma X_i)^{(n-1)!} \cong \pi_{m+1}(\mathrm{Map}_*(S^{n-1}, \bigwedge_{i=1}^n \Sigma X_i))^{(n-1)!}$$

$$\cong \pi_{m+1}(\mathrm{Map}_*(\bigvee_{(n-1)!} S^{n-1}, \bigwedge_{i=1}^n X_i)) \cong \pi_m(\Omega\,\mathrm{Map}_*(\bigvee_{(n-1)!} S^{n-1}, \bigwedge_{i=1}^n X_i)).$$

The first isomorphism follows from the Freudenthal suspension theorem, as each $X_i$ is $k$-connected, so their smash product is $nk$-connected, and so when $0 \leq m \leq (n+1)(k+1)-1$, we have $m \leq 2nk$ (for $k$ at least 1), which is the range where Freudenthal's suspension theorem. The second and second to last isomorphism follow from the adjunction $\Sigma \dashv \Omega$. Lastly, the third isomorphism follows from the fact that $\mathrm{Map}_*(-, X)$ converts coproducts to products and from the fact that $\pi_m$ preserves products.

So we have the desired abstract isomorphism of homotopy groups if we can show that $\Delta_n \simeq \bigvee_{(n-1)!} S^{n-1}$. And it turns out this particular prayer does not go unanswered. In order to obtain the desired weak equivalence, we will need the following result, called the nerve theorem, which we state without proof.

**Theorem 3.4.13.** *(Theorem 21 in [3]) Let $X$ be a CW-complex and assumed it is covered by subcomplexes $K_i$ such that every nonempty finite intersection of these subcomplexes is contractible. Then $X$ is equivalent to the nerve of the poset of finite intersection of elements of $\{K_i\}$.*

**Proposition 3.4.14.** *(Proposition 19 and lemma 20 in [3]) There is a canonical weak equivalence*

$$\bigvee_{(n-1)!} S^{n-1} \to \Delta_n$$

*Proof.* This proof is divided into two key steps. We introduce the space $\widetilde{\Delta_n}$ which is the subspace of $\Delta_n$ consisting of those $t \in \Delta_n$ such that $t_{ij} = 0$ whenever $j > 1$. We will show on the one hand that $\widetilde{\Delta_n}$ is homotopy equivalent to $\Delta_n$ and on the other that $\widetilde{\Delta_n}$ is homeomorphic to a wedge of $n$-spheres. We start with the latter objective. Explicitly the space $\widetilde{\Delta_n}$ is the quotient of $I^{n-1}$, the first row of $I^{n(n-1)}$, by the subspace $A \cap \widetilde{\Delta_n}$, where $A$ is the space by which we quotient $I^{n(n-1)}$ to get $\Delta_n$.
Under this identification, the subspace by which we quotient to get to $\Delta_n$ restrict to the corresponding subspaces of $I^{n-1}$ by which we quotient to get $\widetilde{\Delta_n}$. Referring back to proposition 3.4.8, the subspace by which we quotient $I^{n-1}$ consists of those $t$ such that $t_{1i} = 1$ for some $i$, those such that $t_{1i} = t_{11} = 0$ for some $i$ and finally those $t$ such that $t_{1i} = t_{1j}$ for distinct $i, j$. It is then clear that this is exactly

the configuration space of $n-1$ labeled points in $[0,1]$, quotiented by those configurations where one of the points is 0 or 1. One can observe that, before quotienting, this can be described as

$$\bigsqcup_{\sigma\in\Sigma_{n-1}} \{(s_1, ..., s_{n-1}) \in [0,1]^{n-1} \mid 0 \leq s_1 < s_2 < ... < s_{n-1} \leq 1\}.$$

Of course, the above space is identified with $\bigsqcup_{(n-1)!} \Delta^{n-1}$, and the subspace by which we quotient is $\bigsqcup_{(n-1)!} \partial\Delta^{n-1}$. So we get a homeomorphism $\widetilde{\Delta_n} \cong \bigvee_{(n-1)!} S^{n-1}$.
We now move on to showing we have a weak equivalence $\widetilde{\Delta_n} \simeq \Delta_n$. For this, we introduce some notation for the spaces appearing in proposition 3.4.8. Denote by $Z$ the subspace of $I^{n(n-1)}$ consisting of those $t$ such that $t_{ij} = 1$ for some coordinate. And denote by $W$ the space of $t$ such that for some fixed and distinct $i, j$ and for all $k$ we have $t_{ik} = t_{jk}$. We also denote the corresponding subspaces of $\widetilde{\Delta_n}$ by $\widetilde{Z} = Z \cap I^{n-1}$ and similarly $\widetilde{W} = W \cap I^{n-1}$. So by construction the space of interest to us fits in the following homotopy pushouts

$$\begin{tikzcd} \widetilde{Z}\cup\widetilde{W} \arrow[r] \arrow[d] & I^{n-1} \arrow[d] \\ * \arrow[r] & \widetilde{\Delta_n} \end{tikzcd} \qquad \begin{tikzcd} Z\cup W \arrow[r] \arrow[d] & I^{n(n-1)} \arrow[d] \\ * \arrow[r] & \Delta_n \end{tikzcd}.$$

Thanks to homotopy invariance of homotopy pushouts, it suffices to show $\widetilde{Z}\cup\widetilde{W} \simeq Z \cup W$. In turn these spaces fit in the following homotopy pushouts

$$\begin{tikzcd} \widetilde{Z}\cap\widetilde{W} \arrow[r] \arrow[d] & \widetilde{W} \arrow[d] \\ \widetilde{Z} \arrow[r] & \widetilde{Z}\cup\widetilde{W} \end{tikzcd} \qquad \begin{tikzcd} Z\cap W \arrow[r] \arrow[d] & W \arrow[d] \\ Z \arrow[r] & Z\cup W \end{tikzcd}.$$

It isn't hard to see that $Z, W, \widetilde{Z}$ and $W$ are contractible, so that, for the same reasons as above, it suffices to show $\widetilde{Z\cap W} \simeq Z \cap W$. To do this, we will use the nerve theorem to show that both are homotopy equivalent to $\mathrm{N}(P_n)$ the nerve of the poset of non-trivial partitions of a set of $n$-elements. This will show that $\Delta_n \simeq \mathrm{N}(P_n) \simeq \widetilde{\Delta_n}$.
For this, we denote by $W_{ij}$ the subspace of $W$ consisting of $t$ such that $t_{ik} = t_{jk}$. Clearly $W = \bigcup_{1\leq i<j\leq n} W_{ij}$. We also introduce the notation $\widetilde{W_{ij}} = W_{ij} \cap \widetilde{\Delta_n}$. Of course we have $W \cap Z = \bigcup_{i,j} W_{ij} \cap Z$ and $\widetilde{W} \cap \widetilde{Z} = \bigcup_{i,j} \widetilde{W_{ij}} \cap \widetilde{Z}$, so in order to apply the Nerve theorem to obtain the desired result we require precisely that all the non-empty intersections of a finite number of $W_{ij} \cap Z$ or of $\widetilde{W_{ij}} \cap \widetilde{Z}$ are contractible and that the associated poset is the poset of non-trivial partitions of a set with $n$ elements. We will only treat $W \cap Z$, as the case of $\widetilde{W} \cap \widetilde{Z}$ is perfectly analogous
Recall that a partition is equivalently an equivalence relation, so that to a non-trivial partition $\lambda$ we associate the intersection $\bigcap_{i\sim_\lambda j} Z \cap W_{ij}$. It is clear that any intersection is equal to one coming from an equivalence relation, so that the poset of non-empty intersections is equivalent to the poset of non-trivial partitions, in particular the geometric realization of their nerves are the same. Each of these intersection $\bigcap_{i\sim_\lambda j} Z \cap W_{ij}$ is contractible by a coordinate wise linear homotopy sending $t_{ij}$ to 0 if $i \sim_\lambda j$ and to 1 otherwise. And so the nerve theorem applies to obtain the desired result. $\square$

This shows both the domain and codomain $\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}$ have isomorphic $m$th homotopy group for $0 \leq m \leq (n+1)(k+1) - 1$, and so all that remains to show is that $\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}$ realizes this isomorphism. Also notice that the above proof shows that $\Delta_n = \Sigma^2\,\mathrm{N}(P_n)$ where $P_n$ is the poset of nontrivial partitions of $\{1, ..., n\}$ ordered by refinement. We will not deduce the remaining claim that $\gamma$ realizes the above weak equivalence. According to our main source for this subsection, this would follow from

**Lemma 3.4.15.** *(Proposition 23 in [3]) For each $\sigma \in \Sigma_{n-1}$ there is a map $C_\sigma$ fitting into a commutative diagram*

$$\begin{tikzcd}
\prod_{i=1}^n X_i \arrow[r, "C_\sigma"] \arrow[dd, "q"] & \Omega\,\mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(\Sigma X_1, ..., \Sigma X_n) \arrow[r, "\Omega\phi_{(\Sigma X_1,...,\Sigma X_n)}"] & \Omega\,\mathrm{Map}_*(\Delta_n, \bigwedge_{i=1}^n \Sigma X_i) \arrow[d, "\lambda_\tau^*"] \\
& & \Omega\,\mathrm{Map}_*(S^{n-1}, \wedge_{i=1}^n \Sigma X_i) \arrow[d, "\simeq"] \\
\bigwedge_{i=1}^n X_i \arrow[rr, "\Gamma_{\sigma\tau}\wedge(-)"] & & \Omega^n\Sigma^n(\bigwedge_{i=1}^n X_i)
\end{tikzcd}$$

*for each $\tau \in \Sigma_{n-1}$, where $\lambda_\tau : S^{n-1} \to \bigvee_{\Sigma_{n-1}} S^{n-1} \to \Delta_n$ is the inclusion of the $\tau$ factor. Moreover,* $\deg(\Gamma_{\sigma\tau}) = \delta_{\sigma\tau}$*, where this $\delta$ is the Kronecker $\delta$.*

Assuming that we knew that $\phi$ indeed realized the desired isomorphism, we have

**Theorem 3.4.16.** *(Corollary 14 in [3]) There is an equivalence of $\Sigma_n$-spectra*

$$\partial_n(\mathrm{Id}_{\mathcal{S}_*}) \simeq \mathrm{Map}(\Sigma^\infty \Delta_n, \mathbb{S}).$$

*Proof.* The $\Sigma_n$-equivariant map $\gamma : \mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(-) \to \mathrm{Map}_*([0,1]^{n(n-1)}, \wedge_{i=1}^n(-))$ satisfy the connectivity conditions of corollary 3.4.10 by the above statement, where we claimed that $\phi$ realizes the desired isomorphism. So by proposition 3.4.4 we have

$$\Omega^\infty \partial_n \mathrm{Id}_{\mathcal{S}_*} \simeq \varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} \mathrm{cr}_n(\mathrm{Id}_{\mathcal{S}_*})(S^{k_1}, ..., S^{k_n}) \simeq \varinjlim_{k_1,...,k_n \in \mathrm{N}(\mathbb{N}^n)} \Omega^{k_1+...+k_n} \mathrm{Map}_*(\Delta_n, S^{k_1+...+k_n})$$

$$\simeq \varinjlim_{k \in \mathrm{N}(\mathbb{N})} \mathrm{Map}_*(\Sigma^k \Delta^n, \Sigma^k S^0) \simeq \Omega^\infty\,\mathrm{Sp}(\Sigma^\infty \Delta^n, \mathbb{S}).$$

This proves the desired result. □

This completes the computation of the derivatives of the identity to the level of detail we will pursue. In reality, the equivalence we found $\Delta^n \simeq \Sigma^2\,\mathrm{N}(P_n)$ during the proof of theorem 3.4.13 can be made into a $\Sigma_n$-equivariant equivalence, so that we may write

$$\partial_n(\mathrm{Id}_{\mathcal{S}_*}) \simeq \mathbb{D}(\Sigma^\infty \Sigma^2\,\mathrm{N}(P_n)),$$

where $\mathbb{D}(X)$ is a short hand for $\mathrm{Map}(X, \mathbb{S})$. This is the form used by Prof. Ching to obtain an operad structure on the derivatives of the identity, by identifying the symmetric sequence of nerves of the partition posets with the bar construction on the constant symmetric sequence with value $S^0$ (see lemma 8.6 of [6]). This last symmetric sequence is the operad of commutative algebras of spectra, so that in analogy with the classical case (see for example 6.2 in [8]) the derivatives of the identity can justifiably be seen as a "spectral lie operad".

# A Infinity category vocabulary

In this appendix, we happasardly gather some of the $\infty$-categorical vocabulary required to read this project. We do not claim to be exhaustive, but we hope that this can make reading this project easier. As our goal is simply to gather results, we give minimal intution if any, and don't go into the detail of the definitions required to understand these statements, for those we refer to [20].

In the following definition, when we write $X \star Y$, we mean the join of $X$ and $Y$.

**Definition A.0.1.** (Definition 1.2.8.4. in [20]) Given a simplicial set $K$, then left cone of $K$ is defined to be $K^{\triangleleft} = \Delta^0 \star K$. We define the right cone of $K$ similarly by $K^{\triangleright} = K \star \Delta^0$.

To see that this warrant the name "cone", one can visualize that $(S^1)^{\triangleleft}$ is indeed a cone in the classical sense.

The next three definition are the $\infty$-category analog of some common category theory notions.

**Definition A.0.2.** (Definition 1.1.1.1. in [21]) We call an infinity category pointed if it has a 0 object, i.e. an object which is both initial and final.

**Definition A.0.3.** (Definition 4.3.2.2. in [20]) Let $\mathcal{C}, \mathcal{D}$ be $\infty$-categories and $\mathcal{C}^0$ a subcategory of $\mathcal{C}$. Then we say that $F : \mathcal{C} \to \mathcal{D}$ is a left Kan extension of $F^0 : \mathcal{C}^0 \to \mathcal{D}$ if there is a diagram

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] & \end{tikzcd}$$

such that for every $C \in \mathcal{C}$ the induced diagram

$$\begin{tikzcd} (\mathcal{C}^0_{/C}) \arrow[r, "F^0_{/C}"] \arrow[d] & \mathcal{D} \\ (\mathcal{C}^0_{/C})^{\triangleright} \arrow[ur, "F_{/C}"'] & \end{tikzcd}$$

expresses $F(C)$ as a colimit of $F_C$. This means that $F_{/C}$ can be understood as the initial cocone over $C^0_{/C}$. When $C \notin \mathcal{C}^0$, by $\mathcal{C}^0_{/C}$ we heuristically mean the $\infty$-subcategory of $\mathcal{C}_{/C}$ such that the domain of objects is in $\mathcal{C}^0$. This can be made formal by the fiber product $\mathcal{C}^0_{/C} := \mathcal{C}_{/C} \times_C \mathcal{C}^0$.

We include for convenience the dual definition of right Kan extension.

**Definition A.0.4.** (Definition 4.3.2.2. in [20]) Let $\mathcal{C}, \mathcal{D}$ be $\infty$-categories and $\mathcal{C}^0$ a subcategory of $\mathcal{C}$. Then we say that $F : \mathcal{C} \to \mathcal{D}$ is a right Kan extension of $F^0 : \mathcal{C}^0 \to \mathcal{D}$ if there is a diagram

$$\begin{tikzcd} \mathcal{C}^0 \arrow[r, "F_0"] \arrow[d] & \mathcal{D} \\ \mathcal{C} \arrow[ur, "F"] & \end{tikzcd}$$

such that for every $C \in \mathcal{C}$ the induced diagram

$$\begin{tikzcd} (\mathcal{C}^0_{C/}) \arrow[r, "F^0_{C/}"] \arrow[d] & \mathcal{D} \\ (\mathcal{C}^0_{C/})^{\triangleright} \arrow[ur, "F_{C/}"'] & \end{tikzcd}$$

expresses $F(C)$ as a limit of $F_C$. This means that $F_{C/}$ can be understood as the terminal cone over $C^0_{C/}$. When $C \notin \mathcal{C}^0$, by $\mathcal{C}^0_{C/}$ we heuristically mean the $\infty$-subcategory of $\mathcal{C}_{C/}$ such that the domain of objects is in $\mathcal{C}^0$. This can be made formal by the fiber product $\mathcal{C}^0_{C/} := \mathcal{C}_{C/} \times_C \mathcal{C}^0$.

We need the following result in definition 1.2.2.

**Proposition A.0.5.** *(Proposition 4.3.2.15. in [20]) Let $\mathcal{C}, \mathcal{D}$ be two categories and let $\mathcal{C}^0$ be a full subcategory of $\mathcal{C}$. Let $\mathcal{K} \subset \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ be the full subcategory of those functors which are left Kan extensions of their restriction to $\mathcal{C}^0$ and let $\mathcal{K}' \subset \mathrm{Fun}(\mathcal{C}^0, \mathcal{D})$ be the full subcategory of those functors $F$ such that for each $C \in \mathcal{C}$ the induced diagram $\mathcal{C}^0_{/C} \to D$ has a colimit. Then the restriction functor $\mathcal{K} \to \mathcal{K}'$ is a trivial fibration (of simplicial sets).*

The classical fact that colimits commute with each other (and dually) unsurprisingly holds in the $\infty$-category case, this can be deduced from the following proposition (and its dual).

**Lemma A.0.6.** *(Lemma 5.5.2.3. in [20]) Let $p : X^{\triangleright} \times Y^{\triangleright} \to \mathcal{C}$ be a diagram such that for every vertex $x \in X^0$ the induced diagram $p_x : Y^{\triangleright} \to \mathcal{C}$ is a colimit diagram and similarly that for every $y \in Y^0$ the induced diagram $p_Y : X^{\triangleright} \to \mathcal{C}$ is a colimit diagram. Then, denoting by $\infty$ the cone point of $Y^{\triangleright}$, we have that the induced diagram $p_\infty : X^{\triangleright} \to \mathcal{C}$ is a colimit diagram.*

Another very useful tool to compute colimits is replacement by a cofinal subcategory, which in the infinity category case is dealt with by

**Proposition A.0.7.** *(Proposition 4.1.1.8. in [20]) Let $v : K' \to K$ be a cofinal map of (small) simplicial sets, then if $p : K^{\triangleright} \to \mathcal{C}$ is a colimit diagram for $p|_K$, we have that $p \circ v : K'^{\triangleright} \to \mathcal{C}$ is a colimit diagram of $p \circ v|_{K'}$.*

Yet another tool to compute limits and colimits is the following result.

**Proposition A.0.8.** *(Proposition 4.4.2.2. in [20]) Let $\mathcal{C}$ be an $\infty$-category and let $p : K \to \mathcal{C}$ be a $K$-shaped diagram. Suppose that $K$ decomposes as $K = A \cup B$ and that $p|_A.p|_B$ and $p|_{A\cap B}$ all admit a colimit, say $X, Y$ and $Z$ respectively. Then, the colimit of $p$, can be identified with the pushout $X \cup_Z Y$.*

The following proposition gives us a way to detect whether a functor $L : \mathcal{C} \to \mathcal{C}' \subset \mathcal{C}$ into a subcategory is a localization.

**Proposition A.0.9.** *(Proposition 5.2.7.4. in [20]) Let $\mathcal{C}$ be an $\infty$-category and let $L : \mathcal{C} \to \mathcal{C}'$ be a functor into a subcategory of $\mathcal{C}$, suppose further that this functor is essentially surjective. Then the following are equivalent:*

*(i) $L$ is a left adjoint to the inclusion $\mathcal{C}' \to \mathcal{C}$.*

*(ii) There exists a natural transformation $\alpha : Id_{\mathcal{C}} \to L$ such that for each $C \in \mathcal{C}$, the morphisms $L(\alpha_C)$, $\alpha_{LC} : LC \to LLC$ are equivalences.*

The next result is the $\infty$-category generalization of Quillen's theorem $A$, which will serve us to show that inclusions of certain subdiagrams are final or initial.

**Theorem A.0.10.** *(4.1.3.1. in [21]) Let $f : \mathcal{C} \to \mathcal{D}$ be a map from a simplicial set to an infinity category. Then $f$ is final if and only if for every object $D \in \mathrm{Ob}(\mathcal{D})$, the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible.*

*Dually, $f$ is initial if and only if for every object $D \in \mathrm{Ob}(\mathcal{D})$ the simplicial set $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible.*

Notice that our terminology is different than that of Lurie, see 0.0.8.

Recall that $\mathcal{S}$ denotes a suitable category of spaces. Adding the superscript fin means we are referring only to spaces with the homotopy type of a finite CW-complex. This can also be described as the smallest subcategory of spaces containing a point, stable under equivalence and finite colimits. Adding the subscript $*$ implies we are considering pointed spaces.

In the first chapter of [21], an explicit model for the stabilization of an $\infty$-category with finite limits is given as follows.

**Definition A.0.11.** (Definition 1.4.2.8. in [21]) Let $\mathcal{C}$ be an $\infty$-category with finite limits. The category of spectrum objects of $\mathcal{C}$, or $\mathrm{Sp}(\mathcal{C})$ for short, is the subcategory of $\mathrm{Fun}(\mathcal{S}^{\mathrm{fin}_*}, \mathcal{C})$ consisting of 1-excisive and reduced functors.

For the definition of 1-excisive and reduced we refer the reader to sections §1.1 and §1.3. To see that this category is indeed stable, we refer the reader to corollary 1.4.2.17. in [21], and to observe that it is a model for the stabilization we refer the reader to proposition 1.4.2.24. in [21].
One advantage of this model is that there is an evident map $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ given by evaluating at the 0-sphere $S^0$. The point of this map is highlighted first of all by the fact that because (co)limits are computed pointwise it preserves all (co)limits, and second of all by the following result.

**Proposition A.0.12.** *(Proposition 1.4.2.21. in [21]) Let $\mathcal{C}$ be a category which admits finite limits, then $\mathcal{C}$ is stable if and only if the map $\Omega^\infty : \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence.*

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