Derivatives of K-Theory

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A formula for the derivatives of K-theory

The goal of this project was to use the toolbox offered by *Goodwilie calculus* (notably including a theory of derivatives of functors) to understand the famous (and famously complicated) *Algebraic K-theory* functor. For technical reasons, instead of studying the derivatives of K-theory directly, we study the derivatives of $\widehat{K} : \mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}} \to \mathrm{Sp}$, the functor sending a stable category $\mathcal{D}$ over $\mathcal{C}$ to $\mathrm{fib}(K(\mathcal{D}) \to K(\mathcal{C}))$. Our study culminates in the following formula, whose components we spend the rest of this poster exploring :

$$\partial_* \widehat{K} \simeq \mathrm{coBar}(THH(\mathcal{C}, -^{\otimes *})_{hC_*})), \partial_*(\mathrm{coFree}_{\mathrm{Bimod}(\mathcal{C})}), \mathbf{1}).$$

Disclaimer: There are several minor simplifications in how we present the above formula for increased legibility. Nonetheless, the differences are sufficiently small for the reader to feel safe in considering the above formula as true.

Goodwillie calculus

When studying sufficiently nice functions $f : \mathbb{R} \to \mathbb{R}$, there are many cases where we can replace f by a degree n-polynomial, which is orders of magnitude easier to comprehend. When studying functors $F : \mathcal{A} \to \mathcal{B}$, we might hope for a similar theory, which could replace F by a best approximation of degree n. This theory was first introduced by Goodwillie in a series of three papers starting with [3]. The kicker is that the inclusion of "degree n" functors (in this context often called n-excisive) into the category of all functors admits a left adjoint P_n. And category theory teaches us that $F \to P_n F$ is a perfectly suitable n-excisive approximation.

Goodwillie calculus in some sense goes about differential calculus the wrong way, as from $P_n F$, we define $D_n F := \mathrm{fib}(P_n F \to P_{n-1} F)$, which can be considered the difference between the degree n and degree $(n-1)$ approximations of f, and thus corresponding to $\frac{f^{(n)(a)}}{n!}(x-a)^n$ in classical calculus. And from this, we are able to extract a suitable definition of the nth derivative of F, denoted by $\partial_n F$. Thus we end up defining an analog of nth derivative from an analog nth Taylor approximation.

Perhaps surprisingly since the theory runs in the opposite direction from usual calculus, in practical computations, to study a functor F the program suggested by Goodwillie calculus involves computing the various $\partial_n F$, assembling these to recover the $P_n F$, and then taking the limit over n to define $P_\infty F$, which in good conditions recovers F.

Chain Rule

Whenever one encounters something analogous to a derivative in mathematics, one of the first question one ought to ask is whether there is a chain rule which allows us to understand $\partial_n(GF)$ from $\partial_n F$ and $\partial_n G$. In Goodwillie calculus, this was achieved at the generality needed for this project only recently in [1]. This required assembling all the derivatives into a single object called a symmetric functor sequence (a type of object with a special composition which is the reason for its appearance in the chain rule) and denoted by $\partial_* F$, and is proven by recasting Goodwillie calculus as something quite fundamentally 2-categorical. At this point, the chain rule can be succinctly phrased as saying that for functors $G : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{E}$ with $\mathcal{D}$ there is a natural in F and G equivalence

$$\mu : \partial_*(G) \circ \partial_*(F) \to \partial_*(GF).$$

First we point out that there is a way to obtain a chain rule even when $\mathcal{D}$ is not stable, but it is considerably trickier and did not make an appearance in this project. Second, the presence of an arrow gives the Goodwillie derivatives an algebraic flavor absent from the classical chain rule. The way in which this dimension made an appearance for us is when applied to the right hand side of the following equivalence

$$\partial_*(GF) \to \mathrm{Tot}\, \partial_*(G\Omega^\infty(\Sigma^\infty\Omega^\infty)^\bullet \Sigma^\infty F).$$

Indeed, taking F to be the identity of $\mathcal{D}$ yields the equivalence

$$\partial_*(G) \to \mathrm{Tot}(\partial_*(G\Omega^\infty) \circ \partial_*(\Sigma^\infty\Omega^\infty)^\bullet \circ \partial_*(\Sigma^\infty)).$$

It turns out that Σ^∞ is "linear", and so we can replace its derivatives with the trivial symmetric sequence $\mathbf{1}$. Writing this out explicitly, we spot the so called *cobar construction*, which can be thought of as analogous to free resolutions of modules

$$\partial_*(G) \to \mathrm{coBar}(\partial_*(G\Omega^\infty), \partial_*(\Sigma^\infty\Omega^\infty)^\bullet, \partial_*(\Sigma^\infty)).$$

This turns out to be a *Koszul Duality* statement, which is an equivalence between Algebras and coAlgebras, and their modules and comodules. The "surprise" appearance of this well established algebraic theory comforts us in following the road it suggests to study $\partial_* K$, which is to say rather than study the derivatives of K-theory directly, study the derivatives of $K \circ \Omega^\infty$ and of the comonad $\Sigma^\infty\Omega^\infty$.

Our main formula follows from the above equivalence after identifying $\partial_*(K \circ \Omega^\infty)$ and $\partial_*(\Sigma^\infty\Omega^\infty)$.

Algebraic K-theory

Homotopy theory teaches (or leads us to hope) that when a collection of invariants, say E_0, E_1 and E_2, send a short exact sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ to a long exact sequence

$$\cdots \to E_2(\mathcal{C}) \to E_1(\mathcal{A}) \to E_1(\mathcal{B}) \to E_1(\mathcal{C}) \to E_0(\mathcal{A}) \to \cdots$$

we might expect there to be a functor $E(-)$ into spaces, which sends exact sequences to fiber sequences, and whose first few homotopy groups recovers the invariants E_0, E_1 and E_2. This was the idea which lead to the historical development of K-theory as an invariant of rings, though nowadays K-theory is understood rather as the "universal functor which splits exact sequences of categories" from a certain category of categories to a close relative of the category of topological spaces. This perspective as a universal functor is due to [2], and is the perspective which we used in this project.

To precisely define the domain and codomain of K-theory, we need to indulge in a small tangent. In a homotopic context, the presence of a 0-object and pushouts gives the existence of a suspension functor Σ, generalizing the suspension of topological spaces. In analogy with inverting elements in rings, there is a way to invert $\Sigma : \mathcal{C} \to \mathcal{C}$, which results in the stabilization adjunction

$$\Sigma^\infty : \mathcal{C} \leftrightarrows \mathrm{Sp}(\mathcal{C}) : \Omega^\infty.$$

In case $\mathcal{C} \simeq \mathrm{Sp}(\mathcal{C})$, we call $\mathcal{C}$ stable.

The domain of K-theory is then exactly the category of these stable categories and functors between these which preserve finite limits and colimits. This category is denoted by $\mathrm{Cat}^{\mathrm{ex}}$ and importantly admits a notion of exact sequence, called split Verdier sequences, which K-theory is universal in splitting. The codomain of K-theory is the category of spectra denoted by Sp and obtained by stabilizing the category of topological spaces.

Stabilizing the domain

One of the core features of K-theory is that it is hard, and one of the core lessons of Goodwillie calculus is that stable categories are simpler than unstable ones. For K-theory, since the codomain is stable by design, we are lead to wanting to replace the domain by $\mathrm{Sp}(\mathrm{Cat}^{\mathrm{ex}})$ and thus K-theory by $K \circ \Omega^\infty$. However, in our case this is trivial, as one can observe that the suspension functor is 0 in this case, and so inverting it results in the 0-category. The fix is to replace the domain by $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ (and K by $\widehat{K}$), thus making our first step to study its stabilization. A large part of this is done in [4].

The stabilization of $\mathrm{Cat}^{\mathrm{ex}}_{/\mathcal{C}}$ is the category of so-called $\mathcal{C}$-bimodules, that is to say cocontinuous endofunctors of $\mathcal{C}$. Under this identification, the functor Ω^∞ is identified as sending a bimodule M to the category $\mathrm{Lace}(\mathcal{C}, M)$, whose objects are arrows $x \to Mx$. We are able to propose strong evidence that this category can be identified as coAlgebras for the cofree comonad associated to M, whose existence we prove. The same evidence strongly suggests that the comonad $\Sigma^\infty\Omega^\infty$ is identified in this case with the cofree comonad comonad $\mathrm{coFree}_{\mathrm{Bimod}(\mathcal{C})}$.

Derivatives of $\widehat{K} \circ \Omega^\infty$

The next step in the program suggested by the box on the left is to understand the derivatives of $\widehat{K} \circ \Omega^\infty \simeq \widehat{K} \circ \mathrm{Lace}(\mathcal{C}, M)$, which we write as K^{cyc} for convenience.

We reduce to studying just the first derivative by the equivalence $\partial_n K^{\mathrm{cyc}}(\mathcal{C}, M) \simeq \partial_1 K^{\mathrm{cyc}}(\mathcal{C}, M^{\otimes n})_{hC_n}$ from [6].

Our identification is completed by the Dundas-McCarthy theorem as presented in [5], which gives an equivalence $\partial_1 K^{\mathrm{cyc}}(\mathcal{C}, M) \simeq THH(\mathcal{C}, M)$, where THH stands for topological Hochschild homology and can be understood in a precise sense as the trace of $M : \mathcal{C} \to \mathcal{C}$.

References

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