Belfast Conference on operads and calculus

Dév Vorburger

May 2, 2025

1 Introduction

These are notes I took live during the 2025 ”operads and calculus conference” hosted by Queen's university Belfast, organized by the Issac Newton institute in the context of the ”equivariant homotopy theory in context” program. I want to express my gratitude to everyone who made this conference possible, and to the people who made it possible for me to attend this conference.
The fact that I typeset these notes live, with the main goal being to help me follow along and force me to get distracted. The main goal is therefore NOT for these notes to be reread after the fact. Nonetheless, I have made them available on my website for who knows if they might be of use to anyone?
One consequence of the goal of these notes, and the context in which I wrote them, is that sometimes my typing speed couldn't keep up with what is being said, formal reference and capitalization has been thrown out the window and there are several comments which are observations I made to myself as I was writing. Therefore I want to apologize on the one hand for the quality of these notes, and on the other much more important hand, I want to apologize for any accidental act of disrespect these notes contain, by misspelling names, missing some references and missing some capital letters. If you notice anything which bothers you feel free to send me a mail so that I correct it.

I have tried to separate what was being said by the speaker from my own thoughts by marking the latter in **bold**, though even this will be inconsistent, again due to the nature of these notes.

2 Equivariant and N_∞ operads by Mike Hill

2.1 Lecture 1

Step 1 : Equivariantize everything. Any time there is a diagram, then the group has to act on the diagram (not just the pieces of the diagram). Induction, coinduction, tensor induction, all that jazz.

Step 2 : Equivariant structures will always have two points of view, external/internal form. Internal view e.g. Mackey functor, and external example G-commutative monoids. External : operadic, Internal : easier for computations.

Consider the operad $\mathcal{E}_1(n) = \mathrm{Emb}(\bigsqcup_n I, I)$. The points in $\mathcal{E}_1(n)$ parametrize all ways to compose n loops in a space. And it turns out this fully describes ΩX once we take composition into account (and the Σ_n action on $\mathcal{E}_1(n)$).

To make it a G-equivariant operad, define an operad by replacing the Σ_n action by an action of $G \times \Sigma_n$ (the rest of the definition is the same). (The G-action on the composition law acts diagonally on the left side $\mathcal{O}(n) \times \mathcal{O}(k_1) \times ... \times \mathcal{O}(k_n) \to \mathcal{O}(k_1, ..., k_n)$).

An $\mathcal{O}$-algebra is a space X with maps $\mathcal{O}(n) \times X^{\times n} \to X$ compatible with composition.
The little disks is an orthogonal G-representation V : given by $\mathcal{D}(V)_n = \mathrm{Emb}(\bigsqcup_n D(V), D(V)$.
As the dimension of V increases, the spaces $\mathcal{D}(V)_n)$ become more and more connected.

And so unsuprisingly, try doing it for an infinite dimensional V, you require that every space $\mathcal{O}(n)$ is contractible, and ignoring equivariance, we see that these are Σ_n-free spaces. We call this a $\mathcal{E}_\infty$ space. And this is essentially unique as any choice of a Σ_n-equivariant map $\Sigma_n \to \mathcal{E}_\infty(n)$ gives a multiplication $X^n \cong \Sigma_n \times_{\Sigma_n} X^n \to \mathcal{E}_\infty \times_{\Sigma_n} X^n \to X$. And the fact that this is essentially unique is what gives commutativity.

Now back with G-operads, what if V has a non-trivial (orthogonal) G-action, especially as we go to ∞.

Definition 2.1.1. (Coefficient systems of spaces) Given a Gspace X we have a functor $\underline{X} : \mathcal{O}rb^{G,op} \to$ Top which maps $T \mapsto \mathrm{Map}^G(T, X)$.

Theorem 2.1.2. *(Elmendorf) The assignement $X \mapsto \underline{X}$ gives an equivalence of ininity categories*

$$\mathrm{Top}^G \simeq \mathrm{Fun}(\mathcal{O}rb^{G,op}, \mathrm{Top}).$$

Notice that for $\mathcal{D}(V)_n$ evaluation at the origin of each disc, gives a homotopy equivalence $\mathrm{Conf}_n(V)$. The action restricts as follows, if we have an element (g, σ) it acts on an embeding $f : [n] \to V$ as $(g, \sigma)f(j) = g(f(\sigma^{-1}(j)))$.

Proposition 2.1.3. *The action of Σ_n on $\mathcal{D}(V)_n$ is free.*

THis means we don't care about subgroups of $G \times \Sigma_n$ that intersect Σ_n non-trivially, because the fixed points are trivial.

Proposition 2.1.4. *If $\Gamma \subset G \times \Sigma_n$ has $\Gamma \cap \Sigma_n = \{e\}$, then*

$$\Gamma = \{(h, f(h)) \mid h \in H \subset G, f : H \to \Sigma(n)\}.$$

We call these graph subgroups of $G \times \Sigma_n$.

Remark 2.1.5. A graph subgroup of $G \times \Sigma_n$ is the same as an H-set structure on $[n]$.

Theorem 2.1.6. *Let Γ be a graph subgroup, corresponding to $T \in \mathrm{Set}^G$ of cardinality n then*

$$\mathrm{Cof}_n(V)^\Gamma \cong \mathrm{Emb}^G(T, V).$$

So an algebra for $\mathcal{D}(V)$ is a space with multiplications $\mathrm{Map}(T, X) \to X$ where T is a finite H-space, which are in essence H-twisted maps $X^{|T|} \to X$.

Example 2.1.7. Consider S^1 with the conjugation action denoted S^σ and with the trivial C_2 action, denote it S^1. This gives us two kinds of loop spaces $\Omega^\sigma X$ and ΩX.
We understand the latter relatively well, but for the latter, there is a big question of how to embed two copies of S^σ into S^σ.
And the punchilne is that there isn't really a multiplication map $\Omega^\sigma \times \Omega^\sigma \to \Omega^\sigma X$. But we have twisted multiplications e.g. $\Omega^\sigma X \times (\Omega X \times \Omega X) \to \Omega^\sigma X$.

Definition 2.1.8. An N_∞ operad for G is a G operad such that Σ_n acts freely.
If $\Gamma \subset G \times \Sigma_n$, then $\mathcal{O}(n)^\Gamma$ are either empty or contractible.
The G-fixed points are always contractible.

Example 2.1.9. If U is a universe, then $\mathcal{D}(U)$ is N_∞ and so is the "linear isometries operad" $\mathcal{L}(U)$.

Definition 2.1.10. A finite H-set T is admissible for $\mathcal{O}$ if $\mathcal{O}(|T|)^{\Gamma_T} \neq \emptyset$ for Γ_T any graph subgroup classifying the H-set T.

Then there were examples of admissible sets for several N_∞ operads seen so far, which I didn't type out.

Theorem 2.1.11. *(Blumberg-Hill) If T is any admissible H-set for $\mathcal{O}$ then we have an essentially unique twisted multiplication*

$$\mathrm{Map}(T, X) \to X.$$

Note T being admissible gives a lot of admissible sets, and one can show the collection of admissible sets is a complete invariant.

Definition 2.1.12. (Indexing systems, Blumberg-H) **Too long to type while listening**

But essentially, this is an axiomatization of the poset structure of the collection of admissible sets. And so in general we have a poset of indexing systems $I(G)$.

Definition 2.1.13. (A transfer system) (Rubin, Balchin, Barnes, Roitzheim) A transfer system is a poset structure on subgroups of G, such that:
If $K \to H$ then $K \subset H$
If $K \to H$ and $g \in G$ then $gKg^{-1} \to gHg^{-1}$
If $K \to H$ and $J \subset H$ then $K \cap J \to J$

And the poset of transfer $Tran(G)$ systems is isomorphic to the poset of indexing systems.

Theorem 2.1.14. *(Blumberg Hill) $\mathcal{O} \mapsto \underline{\mathcal{O}}$ gives a fully faithful embedding of ∞-categories $\mathcal{N}_\infty^G \hookleftarrow I(G)$*

Theorem 2.1.15. *(Bonventre-Pereira, Gutiérrez-White, Rubin) Every indexing system is the indexing system associated to an N_∞-operad.*

2.2 Lecture 2

Recall : N_∞ operad same data as indexing or transfer system. They parameterize (uniquely) twisted products $\mathrm{Map}(T, X) \to X$ for admissible T.

Natural question : how are all of these twisted multiplications related ? And what is the "algebraic shadow" of this whole story.
Note that Fin^G is the finite coproduct completion of Orb^G, which has some implications on relating functors out of these categories.

A coefficient system can then be defined as $\mathrm{Fun}^{\prod}(Fin^{G,op}, Set)$. Potentially replacing Set by something else.
If X is a pointed G space, we get a natural coeff system of set, grp, abelian groups

$$\underline{\pi}_k(X)(T) := \pi_k(\mathrm{Map}^G(T, X)).$$

A pay off to enlarging from Orb^G to Fin^G gives an adjunction pair which allows us to restrict, induct and coinduct. And then this gives an adjunction on coefficient systems by pre/postcomposing, though for abstract non-sense reasons who is left who is right is exchanged.

Definition 2.2.1. Given a finite G-set T and a coefficient system for G $\underline{M}$ then define $\underline{M}_T(S) := \underline{M}(T \times S)$ is also a coefficient system.
This lifts the diagram $Fin^G \to \mathcal{C}$ to a diagram to coefficient systems in $\mathcal{C}$, i.e. $Fin^G \to Coeff^{G,\mathcal{C}}$.

Evaluating at a point $G/H \to *$ for any H gets a map $\underline{M}(*) \to \underline{M}_{G/H}(*) = \underline{M}(G/H)$ which is equivariant with respect to the "Weil group of H wrt G", and so factors through fixed points.

If T is an admissible set for $\mathcal{O}$ then $\underline{\pi}_k(X)$ has a T-transfer

$$\underline{\pi}_k(X)_T \to \underline{\pi}_k(X)$$

$$T \mapsto \underline{\pi}_k(X)_T$$

gives a natural definition of a coefficient system. Example of Mackey functor.

Definition 2.2.2. (Lindner category) Consider the category $\mathcal{A}$ the category with objects finite G-sets and morphisms are isomorphism classes of spans. In this category disjoint union is a biproduct.

A Mackey functor is a product preserving functor $\underline{M} : \mathcal{A} \to Set$.

Proposition 2.2.3. *Mackey functors actually factor through commutative monoids.*

Essentially because disjoint union is a biproduct. Also note this means it is natural sometimes to group complete.

Example 2.2.4. The prototypical examples are the representables (group completed by tensoring $\mathbb{Z}\otimes_{\mathbb{N}}$).

Definition 2.2.5. An indexing category for G is a subcategory $Fin^G_{\mathcal{O}}$ that is :
Wide
Finite coproduct complete
Pullback stable.

Proposition 2.2.6. *The poset of indexing categories on G is $Tran(G)$.*

Of course for each of these there is an associated lindner category $\mathcal{A}^{\mathcal{O}}$. And therefore allows to define an $\mathcal{O}$-Mackey functor a a product preserving functor $\mathcal{A}^{\mathcal{O}} \to Set$.

Theorem 2.2.7. *If $\mathcal{O}$ is an N_∞ operad, and X is an $\mathcal{O}$-algebra, then for all k then $\underline{\pi}_k(X)$ is naturally an $\mathcal{O}$-Mackey functor.*

Remark 2.2.8. Notice we have a distribution

$$\bigotimes_I \bigoplus_J X_{i,j} \cong \bigoplus_{f \in \mathrm{Map}(I,J)} X^{\otimes f}$$

where $X^{\otimes f} := \bigotimes X_{i,f(i)}$. To get coherence in the various twisted multiplications, we need to make the above isomorphism equivariant.

Definition 2.2.9. Let $f : S \to T$ be map of finite G-sets, we have a pullback map $f^* : Fin^G_{/T} \to Fin^G_{/S}$. We call a category "LCC" (Locally cartesian closed) if it has both adjoints, left called dependent sum Σ_f and right ccalled dependent product $\prod_f$.
Finite G sets are LCC.

Corollary 2.2.10. *We have a natural transformation $f^* \circ \prod_f \Rightarrow Id$ (I think).*

Then the speaker defined an "exponential diagram" which I did not write down. These diagrams somehow encode distributivity.

Definition 2.2.11. Let $\mathcal{P}$ be the cat whose objects are finite G sets and whose morphism from S to T are $S \leftarrow U \to V \to T$ up to isomorphism. You can think of these as polynomials.
Composition is prescribed in a certain way which I opted not to rewrite.

In this category, the disjoint union is just a product.

Definition 2.2.12. A TNR or Tambara functor is a product preserving functor $\underline{R} : \mathcal{P} \to Set$.

This is a bunch of semi rings and maps between them, sometimes as semi-rings, sometimes as modules over each other, and maybe some others.
Then some variants of $\mathcal{P}$ were introduced which seem quite tricky. Spoke about when two indexing systems are compatible.

2.3 Lecture 3

We are gonna talk about G-symmetric monoidal categories.
Recall the internal definition of a Mackey functor was a product preserving functor out of the liner category of G and the external definition is as extensions of the functor $\underline{M}_\bullet : Fin^{\cong} \to Coeff$ along the inclusion of the core of Fin.

There were then some results about restriction, namely of transfers. The speaker observed that somehow transfer systems are a bit over determined. The collection of the above observations lead to a G-equivariant external perspective on Mackey functors.

So a G symmetric monoidal category can be described as a G coefficient system of symmetric monoidal categories together with a bilinear map $Fin^{\cong} \times \mathcal{C} \to \mathcal{C}$ (I didn't fully get why we need this).

Example 2.3.1. The main example is the coefficient system sending a subgroup H to the category of H sets with disjoint union.
Slightly more generally, a symmetric monoidal Mackey functor is a G-symmetric monoidal category.
Some other examples are G-spaces with cartesian product, or G-representations with tensor product.

A G-symmetric monoidal functor is a symmetric monoidal functor such that it also commutes with the norms.

Example 2.3.2. The homotopy coefficient systems are a good example of this.

I wasn't fully able to follow the talk, but the key idea seems to be that the external perspective works really well to give definitions which aren't too cumbersome. I am going to stop taking notes a bit early for this talk, and try to just follow along.

3 Calculus by Kristine Bauer

3.1 Lecture 1

What is linearity?
She thinks it ll be impossible to take notes... I guess we'll see.
What is "linear" really? And how intrinsic can we make the definition. And then how can we generalize.
Categorically, one choice we can make for linearity is "excision".
Heuristic : different flavours of calculus differ mostly in what it means to be linear.
Now let $\mathcal{C}$ be a category with finite colimits and a basepoint, let $\mathcal{D}$ be a category with finite limits, sequential colimits, basepoint and homotopical.
And let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

We call F excisive if it sends (strict) pushouts to (homotopy) pullbacks.

The key example of smth excisive is homology.

Taking linearity to be excision is a CHOICE, there are other choices (c.f. lecture 3).

We call a functor reduced if it preserves the basepoint. We call it linear if it is reduced + excisive. Note we can reduce a functor the second we have quotients.

A key example is if E is a spectrum then $X \wedge E$ is linear.

Now we construct the linear approximation, for simplicity we may assume it is excisive. Let T be the functor which maps a space X to the pushout diagram obtained by completing $CX \leftarrow X \rightarrow CX$. Then apply F to the commuting square, and replace $F(X)$ by the pullback of what remains. This is TF, i.e. $TF(X) = \Omega F(\Sigma X)$, and of course there is a natural map $t : F \Rightarrow TF$.
We can iterate this and get a (co)tower, and then take the homotopy colimit. Call the resulting map $p_1 : F \rightarrow P_1 F$.

Theorem 3.1.1. $P_1 F$ *is* 1*-excisive.*

Definition 3.1.2. A functor F is stably excisive, if there are constants K, c such that for every pushout $C \overset{g}{\leftarrow} A \overset{f}{\rightarrow} B$ with f k_1 connected and g is k_2 connected, with $k_1, k_2 \geq K$ then the map $F(A) \rightarrow \mathrm{holim}(C \rightarrow B \cup_A C \leftarrow C)$ is $k_1 + k_2 + c$ connected.

Theorem 3.1.3. *If F is stably excisive with constants (K, c), then TF is stably excisive with constants $(K, c+1)$.*

This isn't quite enough to be happy, we also want $P_1 F$ to be universal. And it is a theorem, that (homotopically) this is in fact the case.

3.2 Lecture 2

Recall: we decided to pick excisive +reduced as a notion of linearity. We defined $T^i F \rightarrow T^{i+1} F$ and took the homotopy colimit of the natural diagram.
Define $P_0 F = F(*)$ to be the constant functor. There is a map $P_1 F \rightarrow P_0 F$, and the homotopy fiber of this is $D_1 F$. And $D_1 F$ is linear, whereas in general, $P_1 F$ is just excisive.

Motivation : A degree n polynomial function is one determined by its value on $n + 1$ points. And so we use this as inspiration to define n-excisive.

We work with $\mathcal{C}$ and $\mathcal{D}$ categories which are homotopical, all limits and colimits, in $\mathcal{C}$ sequential colimits commute with finite limits. And we fix $F : \mathcal{C} \rightarrow \mathcal{D}$ a homotopical functor.

Note if F is 1-excisive, it's value at X is determined by its value at two copies of the map $X \rightarrow *$. Kinda like a linear function is determined by its value at 2 points.

And so we can use this heuristic, to call F n-excisive, if its value at X is determined by $n + 1$ copies of the map $X \rightarrow *$.

Remark 3.2.1. Note gluing n-cones CX along the copy $X \times \{0\}$ can be described as the join of X and n points.

The speaker defined an n-cube, Cartesian n-cubes and strongly coCartesian n-cubes.

In a general model category, note we can define the join of X and a set S by

$$\text{"}X * U\text{"} := hocof(\bigsqcup_U X \rightarrow X).$$

Note that infinity categorically, you can do "à la Lurie" and define $X * \emptyset = X$ and $X * \{*\} = \{*\}$ "à la main" and force the value on all other finite sets by "extending by pushout".

Now we can define T_nF by perfect analogy

$$T_nF(X) := holim_{\emptyset \neq S \subset [n]} F(X * S)$$

This comes equipped with a map $F \Rightarrow T_nF$, use this to define.

$$P_nF := hocolim(\cdots \to T_n^k F \to T_n^{k+1} F \to \cdots)$$

The proof that P_nF is in fact n-excisive use a lemma of Goodwillie (with a relatively neat proof of Resk) that

Lemma 3.2.2. *The natural map $F \circ \chi \to T_nF \circ \chi$ factors through a cartesian cube, if χ is an n-excisive strongly coCartesian cube.*

Proof sketch : For each subset $V \subset [n]$, let $\chi_V : 2^{[n]} \to \mathcal{C}$ defined by $\chi_V(U) = \chi(U \cup V)$. This loses a lot of information, to the point that there a LOT of maps which become the identity.
And so we define $Y(V) = holim : \emptyset \neq U \subset [n] \chi_V(U)$, this cube turns out to be desired Cartesian cube through which stuff factorizes. What this cube does, is that it doesn't change much, in fact it is the identity most of the time, except that we replaced the value at the empty set by $TF(X)$? Maybe I misunderstood.
With this lemma, it is easy to see using cofinality arguments that $P_nF(\chi)$ is Cartesian, as desired. Furthermore, this is in fact the universal approximation.

Also notice that n-excisive functors are automatically $n + k$ excisive. And so by universality we get maps $P_{n+k}F \to P_nF$. This gives us a "Taylor tower", and one can hope that this tower converges.

$$F(X) = \mathrm{Holim}(P_nF(X)).$$

Now we can also use the maps $P_{n+1}F \to P_nF$ to take the homotopy fiber, and call this D_nF. Now we call a functor n-reduced if $P_{n-1}F$ is constant equal to the terminal object. A functor which is n-reduced and n-excisive are called n-homogeneous.
It isn't too hard to see that D_nF is n-homogeneous.

Theorem 3.2.3. *For functors from Top to Top we have*

$$D_nF(X) \simeq \Omega^\infty(\partial_n F \wedge_{h\Sigma_n} X^{\wedge n}).$$

But why though? Cannot be told in the remaining 15 minutes, but we can try and do a sketch. There is a 4 way equivalence of categories.

$$n - \mathrm{Homog}(\mathcal{S}_*, \mathcal{S}_*) \simeq n - \mathrm{Homog}(\mathcal{S}_*, \mathrm{Sp})$$

$$cr_n : n - \mathrm{Homog}(\mathcal{S}, \mathrm{Sp}) \simeq \mathrm{SymLinFun}_n(\mathcal{S}_*, \mathrm{Sp}) : \Delta$$

$$cr_n : n - \mathrm{Homog}(\mathcal{S}_*, \mathcal{S}_*) \simeq \mathrm{SymLinFun}_n(\mathcal{S}_*, \mathcal{S}_*) :?$$

$$\mathrm{SymLinFun}_n(\mathcal{S}_*, \mathcal{S}_*) \simeq \mathrm{SymLinFun}_n(\mathcal{S}_*, \mathrm{Sp}) : \Omega^\infty$$

Some motivation, consider the cross effect for functions in general. Won't rewrite this.

Then definition of cross effect for functors.

3.3 Lecture 3

So far we've been talking about 1-kind of functor calculus. In this talk, go through a survey of various calculi.
A quick list :
1. Homotopy calculus.
2. Equivariant calculus tree. (c.f. paper by Prof. Dotto). Linearity is given by G-sets.
3. Orthogonal calcullus. Domain category is euclidean spaces with maps linear iso inclusion. Linearity will be $E(V) \cong holim_U E(U \oplus V)$. And the layers look super similar to those of homotopy calculus. Defined by Michael Weiss, and Greg Arone did some very important computations.
4. Manifold calculus. Studies contravariant functors from poset of opens of a manifold to Top. Linearity is a sort of homotopy sheaf condition. And layers have configuration spaces of the manifold which appears.
5. Abelian calculus. Developped by Brenda Johnson and Randy McCarthy, linear is additive. Layers look an awful lot like the ones from homotopy calculus
6. Dual calculus. Dual to abelian calculus.

Homotopy calculus a bit extra :
What happens for the identity? $P_1(Id) = \Omega^\infty\Sigma^\infty$. And the identity converges sometimes.
Computations of the derivative are due to Johnson. There are connections to v_n-periodic homotopy theory due to Arone-Mahowald, Behrens, Heuts. There are connections to Lie operad and little disks due to Heuts. And chain rule from Arone Ching.

Manifold calculus: Let M be a manifold $\mathcal{O}(M)$ the poset of open subsets and let $F : \mathcal{O}(M)^{op} \to Top$ be a contravariant functor which has some niceness property. Define $T_nF = holim_{V \subset \mathcal{O}_\setminus} F(V)$ where $\mathcal{O}_n$ is the subposet of those opens which are diffeomorphic to a disjoint union of n-discs.
It turns out that T_nF is directly n-excisive in the desired sense.

Some fun facts about Manifold calculus. Let $F = Emb(-, N)$, then if $codim(M, N) \geq 3$, then the tower converge (Goodwillie-Klein-Weiss). In the case $dim(M) = 1, dim(N) = 3$, then finite type knot invariants appear in the tower (Volic). Something about long knots due to Arone-Turchin.

Ket $F : \mathcal{C} \to \mathcal{D}$ be a functor from a pointed category with coproducts to an abelian category. The second cross effect measures the failure to send coproducts to direct sums. For reduced functors we get

$$F(X \vee Y) \cong F(X) \oplus F(Y) \oplus \operatorname{cr}_2 F(X, Y).$$

Definition 3.3.1. Extend F to $F : \mathcal{C} \to Ch(\mathcal{D})$, let $C_2F(X) = \operatorname{cr}_2 F(X, X)$. The functor C_2 is a comonad, which therefore allows you to resolve $F(X)$, getting a bicomplex, take the total complex, and that gives you P_1F. And it turns out this exactly kills off $\operatorname{cr}_2 F$.

This calculus can be relatively clearly dualized. And that gives you dual calculus. It measures the failure of F to send a product to a wedge.

Remark 3.3.2. Notice that cocalculus gives a cotower, and one studies colayers.

Remark 3.3.3. Note the abelian cotower maps to F and F maps to the abelian tower. So one can see how these towers play off of each other, which is especially fruitful in the case where F has stable codomain.
There is a theorem of McCarthy, then vanishing cofiber implies that the tower splits in an interesting way. And this is also the first place where the connection to Tate vanishing is drawn.

In the abelian functor calculus, the connection to classical calculus is particularly nice/clear. In particular chain rules and and directional derivatives.

Definition 3.3.4. Let $\mathcal{C}$ be a category which has an operation

$$f : A \to B \mapsto \nabla f : A \times A \to B.$$

A cartesian differential category is a cartesian category for which ∇ satisfies the seven properties of the directional derivative.

Theorem 3.3.5. *The (homotopy) category of abelian categories is a Cartesian differential category with* $\nabla F(V, A) = D_1 F(A \oplus -)(V)$.

The abelian functor calculus has 2 different types of higher chain rules, though the formulas were too big for me to dare write it all up.

The final horizon, can we talk about calculi in a more general setting? As in more calculi? Yes! There is a general framework by Hess and Johnsonn, which starts with the poset $2^{[n]}$ with strict monoidal product given by union. Let $\mathcal{D}$ be a model category.
A $2^{[n]}$ module is a category $\mathcal{D}$ with an operation $\theta : 2^{[n]} \times \mathcal{D} \to \mathcal{D}$ such that the adjoint $\theta^* : 2^{[n]} \to End(\mathcal{D})$ is monoidal.
They have a theorem which starting with such a $2^{[n]}$-module you can produce a calculus tower, which gives a monad which recovers $P_n F$ in many cases and from a sequence of monads can give a whole calculus tower.

4 Invited Talks

4.1 Marcy Robertson : Configurations of the Punctured Plane, Fibre Sequences, and genus zero solutions to the Kashiwara-Vergne Problem

Want to talk about linearization/rational completion of little disc leads to interesting problems in algebra.
Denote by $Conf_n(X)$ for the ordered configuration point, and $\overline{Conf}_n(X)$ for the unordered configuration.
We have $\pi_1(\overline{Conf}_n(\mathbb{C})) = B_n$ and PB_n in the ordered case.
We have an obvious forget map $u : Conf_{n+1}(\mathbb{C}) \to Conf_n(\mathbb{C})$ is a locally trivial fibration wose long exact sequence in homotopy reduced to a split exact sequence

$$1 \to Free(n) \to PB_{n+1} \to PB_n \to 1.$$

We can turn configuration spaces into an operad, but we can approximate it by the little discs operad.

The operad of parenthesised braids PaB (fundamental groupoid of D_2) is an operad in groupoids.
The groupoids $PaB(n)$ are defined as :
Objects : are parenthesized permutations.
Morphisms : are elements of the braid group
Endomorphism : of a parenthesized permutation is always PB_n

The operad PaB is a cofibrant model for an operad assembled out of fundamental groupoids of $Conf_n(\mathbb{C})$.

Theprounipotent competion of a discrete group G is constructed as
1. Take $\mathbb{Q}[G]$
2. Take the I-adic completion wrt to the augmentation ideal $\mathbb{Q}[G]^\wedge$.
3. Let $\hat{G} = grp(\mathbb{Q}[G]^\wedge)$.

Prounipotent groups have an associated lie algebra

For the free group the associated lie algebra is something constructed from the free lie algebra.

For the pure braid group PB_n, then the associated lie algebra to the unipotent compéetion is the Drinfeld-Kohno Lie algebra.

Using this, we can be tempted to replace the (endo)moprhisms of $PaB(n)$ by the unipotent completion of PB_n, giving access to some lie theory. Call the resulting operad $PaCD$. In particular we get access to something called Dringeld associators. These classify object fixing isomorphisms of operads between $P\hat{a}B$ and $PaCD$.
This allows to define quantization functors (turn a symmetric monoidal category into a braided category).

Definition 4.1.1. A symmetric monoidal structure on $\mathcal{C}$ is called infinitesamaly braided if I can define structure maps $t : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ such that ...

It turns out $PaCD$ is the universal infinitesimally braided category. And we get that infinitesimally braided monoids categories are algebras over $PaCD$. Similarly braided monoidal categories are $P\hat{a}B$-algebras.

A tangential derivation t of lie_n is a derivation which is conjugation by some (potentially distinct) element on each generator of lie_n (the free lie algebra on n-generators.) Notice that it is specified by a choice of word for each generator, by whcih t conjugates said generator.
If such a derivation is an automorphism we call it a tangential automorphism. We call the group of tangential automorphism for lie_n by $TAut_n$.

A solution to the Kashiwara-Vergne conjecture or KV-solution is a tangential automorphism $F \in TAut_2$ such that

$$F(e^{x_1} e^{x_2}) = e^{x_1 + x_2}.$$

And the Jacobian satisfies some complicated identity.
In 2012, Alekseev and Torossion showed that the set of KV-solutions is non-empty.
You can use Drinfeld associators can be used to construct KV-solutions.
KV-solutions govern a certain quantization problem (that I didn't quite have the time to understand).

Definition 4.1.2. Given an operad P, we call M a P-moperad if it is a monoid in the category of right P-modules.

Example 4.1.3. By shifting degree, any operad P has an associated P-moperad $P[1]$.

Example 4.1.4. Another, more complicated example, is $D_2^1(n)$ the space of embeddings of n 2-discs in a cylinder. This a $D(\mathbb{R}^2)$-moperad. Where the operad action is by analogy with the composition structure of the 2-disc operad, and the monoid multiplication is stacking of cylinders and rescaling.

Now we construct a PaB-moperad, denoted PaB^1, which we obtain by taking fundamental groupoid of the previous $D(\mathbb{R}^2)$-moperad. The groupoids $PaB^1(n)$ are defined by
Objects are parenthesized permutations of $\{0\} \cup [n]$ which fix 0.
And the automorphism groups of objects are all $PB_n^1 \cong PB_{n+1}$ (the isomorphism is of groups, cannot be lifted to something topological/operadic).
Note that PB_n^1 is the fundamental group of $Conf_n(\mathbb{C}^\times)$.

Then defined map of moperads (potentially over distinct operads) in the "obvious" way.

Theorem 4.1.5. *There is a one-to-one correspondence between KV sokutions constructed from Drinfled associators and object fixing moperad isomorphisms $PaB^1 \to PaCD^1$.*

Once I have such an isomorphism, pulling back along this gives a functor from algebras over the latter to the form, which is to say from braided monoidal cateogries to infinitesimally braided categories (or the opposite I didn't quite follow). Which is a "quantization" process.

4.2 Francesca Prateli : A un/straightening equivalence for ∞-operads

§0 motivation

Definition 4.2.1. Consider $Left_{\mathcal{C}}$ the category of left fibrations over $\mathcal{C}$, i.e. ∞-category over $\mathcal{C}$ which is cofibrant over $\mathcal{C}$. I.e. maps in $\mathcal{C}$ induce maps between fibers.

Theorem 4.2.2. *Let $\mathcal{C}$ by an ∞-category and $\mathcal{S}$ the category of spaces, there is an equivalence*

$$St^{\mathcal{C}} : Left_{\mathcal{C}} \leftrightarrow Fun(\mathcal{C}, \mathcal{S}) : UnSt^{\mathcal{C}}$$

Want to replace ∞-categories by ∞-operads.
Then space valued functors, becomes space valued algebras, and we are going to replace left fibrations by operadic left fibrations. What this will mean is that over a multimorphism $(c_1, ..., c_n) \to d$ induces a morphism between the fibers $\mathcal{C}_{c_1} \times ... \times \mathcal{C}_{c_n} \to \mathcal{C}_d$.
One problem is that there are many models for infinity operads. There will be two models in this talk, Lurie's model and complete dendroidal Segal spaces (which we know to be equivalent). And even an equivalence of models doesn't automatically give equivalent theories of operadic left fibrations, algebras or of the straightening/unstraightening equivalence.
Want to construct a new equivalence un/straightening equivalence

$$left_{\mathcal{O}}^{operad} \cong Alg_{\mathcal{O}}(\mathcal{S})$$

using the symmetric enveloppe functor.

§1.1 un/straightening of Lurie's operads via symmetric monoidal envelope.

Definition 4.2.3. There is an "obvious" forgetgul functor from sym. monoidal. ∞-categories to ∞-operads. There exists a left adjoint called the symmetric monoidal enveloppe.

Proposition 4.2.4. *(Hinnich) We can restrict the un/straightening equivalence for an ∞-symmetry monoidal category $\mathcal{C}$ seen as ordinary categories to an equivalence of symmetric monoidal left fibrations and lax functors $\mathcal{C} \to \mathcal{S}$.*

The speaker further restricted to strong symmetric monoidal left fibrations and strong monoidal functors $\mathcal{C} \to \mathcal{S}$.

§1.2 Operadic left fibration

Definition 4.2.5. A map of ∞-operads is called a left operadic fibration if it is a left fibration after considering these ∞-operads as ∞-categories.

The speaker proved that symmetric monoidal enveloppe gives an equivalence between the category of operadic left fibrations and $\mathcal{O}$-algebras in spaces.

Okay how about the other model?

§2 Model invariance of left fibrations of ∞-operads §2.1 Dendroidal model The dendroidal category is a certain category of trees, equipped with a partial monoidal structure whcih corresponds to grafting trees. Use it to define dendroidal sets similarly to how you define simplicial sets.
Notice that any tree defines a free discrete operad. Similarly to how any element of the simplex category defines a discrete category (the poset $[n]$).
We define a dendroidal ∞-operad as a presheaf (of spaces) on the dendroidal category satisfying a sort of Segal condition (i.e. that if a tree is obtained by grafting, then the value of the operad on that tree is determined by the value on the grafted pieces). We also require that when viewed as a simplicial set (via restriction), it is a complete simplicial set.
§2.2 Dendroidal left fibration

Definition 4.2.6. A map between Dendroidal infinity operads is called a dendroidal left fibration if it is local with respect to the inclusions of leaf horns of T into T for any tree T.

There is an un/straightening in this context.

§2.3 Hiinich and Moerdijk (and Heuts?) showed that there is an equivalence between two categories of left operadic fibrations we have been discussing (I think i kinda got lost at this point). This equivalence is shown via a functor between $\Delta/FinSet$ and the closure of the dendroidal category under direct sum

4.3 Natalie Stewart : Homotopy-coherent interchange and equivariant little disk operads

In trace methods for real algebraic K-theory, THH has a real analogue, called THR, which takes a ring with an anti-involution, and spits out a C_2 spectrum. And there is also a "real" trace map from real algebraic K-theory to THR. This is neat because it is also related to L-theory.
Now a natural question is if I plug in a highly structure ring spectrum with anti-involution in THR, do I get some fun/fancy/interesting multiplication.
Classically, i.e. for THH, we know that it takes an $\mathbb{E}_n$ ring it spits out a $\mathbb{E}_{n-1}$ ring. This can be seen by making THH symmetric monoidal in a way, and with the Boardmann Vogt tensor product

it turns out you can do something similar C_2 equivariantly. In order for the proof to go through, one would hope that $\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$, where V and W are orthogonal G-representations, $\mathbb{E}_V$ is the associated little disc operad, and the tensor product is Boardmann Bogt. And it turns out this is true. This allows to prove that THR of an an $\mathbb{E}_{V\oplus\sigma}$-algebra is a $\mathbb{E}_V$-algebra.

Non-equivariant formulas of the form $\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$ have a rich history (notably see DAG VI for Lurie's proof of the non-equivariant case).

$G-\infty$-operads also have a rich history. Combining these two rich histories together, one expects at some point in history for the formula

$$\mathbb{E}_V \otimes \mathbb{E}_W \simeq \mathbb{E}_{V\oplus W}$$

to hold equivariantly. And I think this is what the speaker proved, but I didn't quite catch the generality at which it was done.

The talk then talked about G-operads, and I wasn't able to type whilst trying to follow along.

4.4 Connor Mallin : Semiadditive geometry and classifications of Goodwillie towers

Recall that $Sp(\mathcal{C})$ can be thought of as the tangent plane to $\mathcal{C}$. We work with the extra assumption that $\mathcal{C}$'s stabilization is a localization of the category of spectra.
We consider functors $F : \mathcal{C} \to Sp(\mathcal{C})$, and the associated Taylor tower.
The category of polynomial functors $Poly(\mathcal{C}) := Poly(\mathcal{C}, Sp(\mathcal{C}))$ is the category of functors which are n-excisive for some n. And recall alg geom inutition that this is very geometric. Because polynomials.
We call $\mathcal{C} \cong_P \mathcal{D}$ polynomially equivalent if we have a monoidal equivalence of categories of polynomial functors.
The tangent category of $\mathcal{C}$ is an invariance of polynomial equivalence, justifying viewing this as a "geometric" notion under the above intuition.

Recall operads appear naturally when taking Goodwillie derivatives.

There is a very general classification of polynomial functors given by

$$Poly(\mathcal{C}) \cong CoAlg^{<\infty}(\partial_* \Phi).$$

Where Φ is the right adjoint to ∂_*, and this adjunction is comonadic.
This is quite general, so there is a lot to gain by going in a specific case, i.e. in the case of spaces we get "divided power right lie module structures on the derivatives".

In general, bounded right modules over $\mathcal{O}$ or right divided power modules classify only some polynomial functors, as they are included (usually properly) in $CoAlg^{<\infty}(\partial_* \Phi)$.
We can wonder if sometimes the inclusion is not proper.

We call a stable ∞ category 1-semiadditive if the norm map $X_{hG} \to X^{hG}$ is an equivalence for all finite groups, or equivalently all tate constructions X^{tG} vanish for finite groups.

We call a ∞-category is infinitesamally semi-additive if its stablisation is semi-additive.

Proposition 4.4.1. *TFAE:*
1. $\mathcal{C}$ is infinitesimally semi-additive.
2. There is sym mon equivalence of $(Poly(\mathcal{C}), \wedge)$ and bounded right O modules under day convolution.
3. The derivative maps is an equivalence? From like polynomial functors to bounded symmetric sequences which are right $\partial_* Id$-modules.
He went really quick, wasn't able to catch what was written

Definition 4.4.2. The ∞-category of right comodules over an operad $\mathcal{O}$ in Sp_E is $\mathrm{Fun}_{\mathrm{Sp}_E}(Env(\mathcal{O}), \mathrm{Sp}_E)$. A divided power right $K(\mathcal{O})$ module is the ∞-category of coalgebras for the comonad $Indec_{\mathcal{O}} \circ Triv_{\mathcal{O}}$.

Theorem 4.4.3. *There is an equivalence*

$$Poly(\mathcal{S}_*, \mathrm{Sp}) \cong RMod_{lie}^{dp,\infty}.$$

And explicitly this divided power structure is obtained via an equivalence $\partial_* F \simeq indec_{com}(\epsilon(F|_{FinSet_*})$.

Proof idea: prove it for representable functors, and then Kan extend. An important tool in the proof relies that finite sets include into Top and vanishing of tate constructions on derivatives of representables.
The former property doesn't generalize well, but the latter does, and we take that as definition of "differentially semiadditive".

Theorem 4.4.4. *TFAE:*
1. $\mathcal{C}$ is "differentially semiadditive"
2. $Poly(\mathcal{C}) \simeq$ derived power bounded right modules for some operad.
3. The derivatives have a lift into right modules divided power bounded over derivatives of the identity and this llift is an equivalence.
4. ??
He went fast again, wasnt able to catch everything
Then the speaker talked about "significant Tate vanishing" and was able to show that derivatives of the identity having significant tate

vanishing is equivalent to $\mathcal{C}$ being "differentially semiadditive".

Proposition 4.4.5. *Algebras for some operads in Sp_E is differentially semiadditive" if and only if Sp_E is semi additive.*

Then the speaker spoke about "Snaith type localizations" which I wasn't able to fully follow. And then spoke of the "geometry" of Snaith type localizations.

4.5 Muriel Livernet : Multicomplexes and Spectral Sequences: a Homotopy point of view

Let R be a ring and $SpSe$ the category of spectral sequences on that ring.

Remark 4.5.1. The category of spectral sequences is an additive category, but not complete or cocomplete.

Definition 4.5.2. A map $f : A \to B$ in $SpSe$ is an E_r equivalence if it is a quasi-isomorphism from page r onwards.

This gives a category with weak equivalences, with all isomorphisms, two out of three property and some more stuff, called $(SpSe, E_r)$.

There then was a slide with a pretty big diagram with a bunch of "origins" for spectral sequences essentially. In particular some functor $E : \mathcal{C} \to SpSe$, one can pullback weak equivalences along any such E and then ask:
1. Is there a model structure on $\mathcal{C}$ with these pullbacked weak equivalences?
2- Could it be cofibrantly generated?
3. What happens for different r?
4. etc.

Then recalled the projective model structure on unbounded chain complexes.
We would like to do the same with filtered chain complexes.

The speaker then defined a multicomplex, which is a lot of data, so I did not copy the definition. However note that there is a notion of N-truncation, and the case $N = 2$ returns the theory of bicomplexes.

Remark 4.5.3. They seem to be similar to spectral sequences, but will all the data on a single page?

We then spoke about a generalization of the totalization functor to multicomplexes.

Multicomplexes and filtered complexes do actually admit a right proper cofibrantly generated model structure for each r whsoe weeak equivalences are pulled back from r-weak equivalence of $SpSe$.

Proposition 4.5.4. *The category of multi/filtered complexes is abelian.*

Proof sketch : find a ring over which these are modules.

This allows to construct a restriction/induction type adjunction, which turns out to be a Quillen equivalence.

N truncated multicomplexes admit an involution $(-)^{op}$ and taking the associated spectral sequence $E(A)$ then gives two spectral sequences associated to A, i.e. $E(A)$ and $E(A^{op})$.

Now to compare model structures, it seems as though this involution is a bit annoying, and so you really work with the point of view that multicomplexes are modules over a ring.

One special case of all the above, is that there is an adjunction between 2 and 4-truncated multicomplexes, which is relevant in (almost) complex geometry.

Another direction to fix spectral sequences not having themselves a model structure, instead of pulling back to multicomplexes, but instead to pushforward to "extended spectral sequence", i.e. spectral sequences such that there is a map from the $r+1$th page to the homology of the rth page, which isn't necessairly an isomorphism.

The category of extended spectral sequences is much better, namely complete and cocomplete.

What this can be used to do is to provide infinity category models for spectral sequences as relative categories (in the sense of Barwick-Kan), because spectral sequences are homotopically full in extended spectral sequences.

4.6 Sofia Martinez Alberga : Transfer systems and model structures

Recall : Transfer systems allow you to understand equivariant analogues of higher coherence.
Recall : The definition of model structure, which I am not recalling.
Recall : A transfer system in a category is a wide subcategory closed under pullback along any morphism in the ambient category.
Note : Acyclic cofibrations are transfer systems (I think, went by a wee bit quick).

Remark 4.6.1. If you have a model structure on a lattice/a poset, then a composition is a weak equivalence if and only if each morphism is a weak equivalence. (I think)

One way to construct new model structures on a fixed category is by left/right localizing. Right localization wants to fix fibrations/acyclic cofibrations and left localization wants to fix cofibrations and acyclic fibrations. Both want to increase weak equivalence, and therefore both have to increase the classes of maps they don't fix.

Theorem 4.6.2. *(BOOR, 23) Every model structure on a total order lattice can be obtained via a sequence of left and right localizations starting at the trivial model structure.*

Inspired by this the speaker (and others) gave, for any lattice, an explicit description of the set of morphism which induce a model structure coinciding with right localization at a morphism.

And the question is again can we get more general? No... For C_{pq^2}, there are many model structures on the poset of subgroups, that don't come from successive left/right localization from the trivial model structure,
The problem with the model structures which cannot be reached are not saturated.

4.7 Danika Van Niel : How compatible pairs of transfer systems witness equivariant structure

Think of G-Tambara functors as equivariant analogs for commutative rings. It is a lot of data, even in a simple case where $G = C_6$.
Biincomplete G-Tambara functors is something more general, controlled by the combinatorics of compatible pairs of G-transfer systems.

A G trasnfer system is a poset $K \to H$ structure on the set of subgroups of G such that
1. $K \to H$ implies $K \leq H$
2. Reflexivity
3. Transivity.
4. If $K \to H$ and $L \leq G$, then $K \cap L \to H \cap L$.

Example 4.7.1. The poset consisting of all incluions, the complete transfer system, s in fact a transfer system.

Definition 4.7.2. A transfer system whose edges satisfy that if $L \to K$ and $L \to H$m then $K \to H$ is called saturated.

We call the hull of a transfer system, the minimal saturated transfer system which contains the starting transfer system.

Definition 4.7.3. A pair (T_1, T_2) of G-transfer systems are compatible if $T_1 \subset T_2$ and a mildly technical condition which I won't copy.

Example 4.7.4. A transfer system T is always compatible with its hull and with the complete transfer system.
If these are the only one, then we call T a LSP transfer system.

An easy result is that a connected transfer system is LSP. Are there non easy examples?
It turns out that yes, there are transfer systems with two connected components which are LSP, however, that's it!

4.8 Thomas Blom : The chain rule in Goodwillie calculus

The chain rule tells you how to compute derivative of a composite, and then you have the product rule, and ofc put these together and you get "Faà di Bruno's formula" for the nth derivative of $f \circ g$.
We work with presentable pointed ∞-category and that filtered colimits commute with finite limits. We call such an ∞-category differentiable.
Recall the relation between $\partial_n F$ and $D_n F$. We want to prove a chain rule for $\partial_n F$.
Write $\mathrm{SymFun}(\mathcal{C}, \mathcal{D})$ for the category of symmetric sequence of functors, we add superscripts for certain properties in each variable : $*$ for reduced, ω for finitary and L for preserves colimits.
The derivatives define a functor

$$\mathrm{Fun}^{*,\omega}(\mathcal{C}, \mathcal{D}) \xrightarrow{\partial_*} \mathrm{SymFun}^L(Sp(\mathcal{C}), Sp(\mathcal{D})).$$

Taking $\mathcal{C} = \mathcal{D}$ boh sides have a monoidal structure, so we can ask/want that ∂_* has some monoidality of some kind. In general, we'll get lax monoidality.
The classical strategy involves using smth for a left adjoint, but the derivative does not have a left adjoint.
However, we can factor ∂_* as a composition $P_1 \circ \mathrm{cr}_\bullet$, and P_1 is a localization and $\mathrm{cr}_\bullet$ has a left adjoint. So we can do some stuff because for these there are tools to proove monoidality.
However, when we factor it as suggested, the middle object is $\mathrm{SymFun}^{*,\omega}(\mathcal{C}, \mathcal{C})$ and this doesn0t have an associative composition product very dissapointingly.
So we'll need to use an extra detour, by replacing $\mathrm{SymFun}^{*,\omega}(\mathcal{C}, \mathcal{C})$ by $\mathrm{SymFun}^L(\mathcal{P}^{fin}\mathcal{C}, \mathcal{P}^{fin}\mathcal{C})$.
This is related to the original category by an adjoint, and this category has an associative composition product. Using this detour, we can prove the desired chain rule, by showing that via the detour, the functors are sufficiently monoidal.

Now if $\mathcal{C}$ is differentiable, then in fact the derivative is in fact STRONG monoidal, thus giving a chain rule.
And in fact! Unstably, you don't get a chain rule, as can be seen by considering what happens for the identity in spaces.
But we can kinda fix this issue, by replacing composition product by a "relative composition product" using the bar construction. The lax structure gives you a map $\partial_* G \circ_{\partial_*} \mathrm{Id}\, \partial_* F \to \partial_*(G \circ F)$.
One might hope that this is map is an equivalence, i.e. although we don't have a chain rule for composition product, we do for a corrected composition product. And it turns out we do! Yey!
The key ingredients in the proof, is to reduce to the stable setting, where we have a chain rule, and use the Bousfield-Kan resolution and Koszul duality to do this somehow.

As an application, you can take $\mathcal{C}$ to be spectra, in which case the category $\mathrm{SymFun}^L(\mathcal{C}, \mathcal{C})$ is in fact symmetric sequences without any arity 0 stuff. And so (co)algebras in this category (in particular derivatives of (co)algebras in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$) give (co)operads. In particular doing this for the identity functor, which is of course an algebra. One can show in this case one gets something Koszul dual to

the non unital cocommutative operad. So deserves to be called the spectral lie operad.
There was then an application combining the above with Snaithe splitting, which related the spectral lie operad to the $\mathbb{E}_n$-operad, in particular there is a map $Lie[n] \to \mathbb{E}_n^{nu}$ from the shifted lie operad to the non-unital $\mathbb{E}_n$ operad. One kinda wants to say heuristically that this is a map that corresponds to ”envelopping algberas”.
Another application, is that this technology gives ”wrong was maps” $\mathbb{E}_{n+1}^{nu} \to \mathbb{E}_n^{nu}[1]$

4.9 Guy Boyde : Classifying vector bundles

What does calculus have to do with classifying vector bundles?

Let $Vect_d(X)$ be complex vector bundles of rank d over X. seen as a functor, it is corepresentable by $BU(d)$. Direct suming a vector bundle with a copy of the rank 1 trivial vector bundle gives a stabilization map $Vect_d(X) \to Vect_{d+1}(X)$, and you can go to ∞ and this is still corepresented by BU.
There is a natural inclusion map $U(d) \to U(d+1)$ which fits into a fibration

$$U(d) \to U(d+1) \to S^{2d+1}$$

One can use this to see that when $2d \geq dim(X)$ we have $[X, BU(d)] \cong [X, BU(d+1)]$, which recall has an interpretation in terms of vector bundles.

Now this was all relatively classical, calculus has something to say about $Vect_d^0(X)$ which is to say those vector bundles of rank d which are stably trivial.

Remark 4.9.1. A paper by Atiyah and Reese was reccomended

We have a fibration

$$\Omega BU \to fib \to BU(d) \to BU.$$

Apply $[X, -]$ then we get

$$\tilde{K}^{-1}(X) \to [X, fib] \xrightarrow{f} [X, BU(d)] \to [X, BU]$$

This is exact, and we care about the kernel of the last, i.e. we care about the image of f.
View $BU(-)$ as a functor from complex vectors to spaces, and use unitary calculus to write $P_n(BU(-))$.
For example $P_0(BU(-)) = BU$.
Write $D_{[1,\infty]}F := fib(F \to P_0 F)$.
Then in a range we have

$$[X, D_{[1,\infty]}F(V)] \cong [X, D_1 F(V)] \cong [X, \Omega^\infty \mathbb{D}_1 F(V)] \cong [\Sigma^\infty X, \mathbb{D}_1 F(V)].$$

Taking F to be BU, we see the left most thing is what we care about (stably trivial vector bundles), and the right most thing can be attacked with pure stable homotopy theory, and the computation has been done.
This is good enough for there to be some pretty neat computations to pop out.

The speaker and Niall Taggart want to use realification to extend the above results to real vector bundles.

The slides then got quite hard to follow and type out at the same time.

4.10 Gregory Arone : Polynomial functors from the category of groups to a stable infinity category

Let Fr be the category of finitely generated free groups. Let $\mathcal{D}$ be a stable infinity category. We want to understand $[Fr, \mathcal{D}]$

Let $B : Fr \to \mathcal{S}_*$ be the classifying space functor. It induces a restriction functor $\rho_B : [\mathcal{S}_*, \mathcal{D}] \to [Fr, \mathcal{D}]$. This is not at all an equivalence, but restricting can give an equivalence and it does.

Theorem 4.10.1. *Restrict and corestricting ρ_B gives an equivalence*

$$Exc_n(\mathcal{S}_*, \mathcal{D}) \to Poly_n(Fr, \mathcal{D}).$$

Definition 4.10.2. A cubical diagram is split strongly coCartesian, if the maps $\chi(\emptyset) \to \chi(\{i\})$ are split inclusions.

We call a functor polynomial if its $n+1$th cross effect vanishes, i.e. in the case of stable codomain if it preserves split strongly coCartesian diagram. Clearly n-excisive implies n-polynomial, but the other inclusion is false in general.

The inverse to $Exc_n(\mathcal{S}_*, \mathcal{D}) \to Poly_n(Fr, \mathcal{D})$ is the composition

$$Poly_n(Fr, \mathcal{D}) \xrightarrow{l} Exc_n(\mathcal{S}_*^1, \mathcal{D}) \xrightarrow{R} Exc_n(\mathcal{S}_*, \mathcal{D}).$$

Where the first map is left Kan extension along B into connected spaces, and the second is right Kan extension along the inclusion of connected space into spaces.
This is nice, because we know $Exc_n(\mathcal{S}_*, \mathcal{D})$ quite well bc of

Theorem 4.10.3. *The following restriction is an equivalence*

$$Exc_n(\mathcal{S}_*, \mathcal{D}) \to [FIn_*^{\leq n}, \mathcal{D}].$$

And this functor category is also equivalent, by taking the cross effect, to $[Epi^{\leq n}.\mathcal{D}]$ (by a result of Proshvili, Holmstuter and Walde).

This is nice because the catgeory of finite sets and surjections is simpler than the category of free groups.
Also this admits an operad interpretation, $[Epi^{\leq n}, \mathcal{D}]$ can be interpreted as n-truncated right comodules in $\mathcal{D}$ over the non-unital commutative operad, which by Koszul duality is the same as the category of divided power right modules over Lie.
So we compose a bunch of equivalences and get

$$Poly_n(Fr, \mathcal{D}) \simeq Com - comod_{\leq n}(\mathcal{D}) \simeq DP - Lie - Mod_{\leq n}(\mathcal{D}).$$

So that's a discussion of the general result, now we talk about applications.

Consider the category $[Fr, Ab]$. For example, there is abelianization $ab(G)$, or n-th tensor power of abelinization $T^n \circ ab$, or n-th exterior power or ...
How abt an example where it doesn't come from abelinization? Let IG be the augmentation ideal, and define $Pa_n(G) = IG/IG^{n+1}$, this is called a Passi functor and it comes up sometimes as projective objects of $[Fr, Ab]$.

It turns out all of these are polynomial functors.

Of course $[Fr, Ab]$ is an abelian category, so we could want to compute Ext-groups.
For example, Vespa showed in 2018

$$Ext^*_{[Fr,Ab]}(T^m \circ ab, T^n \circ Ab) \cong \Sigma^{n-m}\mathbb{Z}(Sur(n.m)).$$

There are other computations, such as rational ext group for exterior powers, and of Passi functors with tensor powers.

Using the new techniques the speaker introduced, some more precise computations, for instance Ext groups between 2nd exterior power to the nth exterior power.
Sketch of the method using these new tools :
By general abstract non-sense

$$Ext^*_{[Fr,Ab]}(F,G) = \pi_{-*}(\underline{Nat}(F,G)) = \pi_{-*}(\underline{Nat}(\hat{F},\hat{G}))$$

Where the middle term means we have passed to derived functors to the stable category of chain complexes and $\hat{F}$ means the functor $Fr \to \mathcal{S}_*$ corresponding to F under the equivalence discussed above. And so now we are computing natural transformations of functors to $\mathcal{S}_*$. And here we can expect to find some fun stuff because we have access to all of topology, which one can believe is easier when it comes to computing homology. Especially because we can hope that some of our functors are representable.

A cool application of the application is to compute stable homology groups of $Aut(F_n)$. A while back Friedlander and Suslin used polynomial functor methods to say something about stable homology of GL_n.
You can use similar ideas for $Aut(F_n)$ with the tools discussed above.

4.11 Scott Balchin : Generation combinatorics of G-transfer systems

Methods for counting G-transfer systems, and can we use this as a group invariant.
Recall the definition of a G-transfer system.

What are some algorithms to check this number?
Brute force on all lattices $Sub(G)$, which requires a lot of checks, like it is a deeply demanding computation.
You can go by recursion, and try breaking up your lattice, which has some success.
Most lattice operations don't mesh well with counting all transfer systems, which is very disappointing. But there are some successes, namely the fusion product.
The last theoretical trick, is something called lossless, which is property which allows to quotient out by an action of conjugacy on the various transfer systems to do something.

But this is limited, even for S_4, none of these methods particularly work.
But some less theoretical trickery, but like algorhitmic trickery can do something. Namely something called "Rubin's algorithm". Which is a way, starting from a single inclusion, generates a minimal transfer system.

Recall the definition of a closure operator.

Lemma 4.11.1. *Rubin's algorithm is a closure operator for the set of lattice structures on the subgroups of a fixed group. And the closed sets are exactly the transfer system.*

And we actually have pretty good algorithms for finding closed sets using a closure operator, in fact we have an "embarassingly parallel" algorithm. This algorithm sort of runs by induction, generating some transfer systems at step 1, some at step 2, etc.
This gives a rank function the collection of transfer systems.

The fact that we have an algorithm that sends a set of inclusions to a transfer system allows us to ask what are the minimal generating sets for a given transfer system.
Although minimal generating sets are not unique, they do all have the same size. And this gives a rank function on the collection of transfer systems. And it turns out this is the same as the rank function from earlier. Call it $m(-)$

We can use this rank function on transfer systems, to obtain group invariants. For example, define the width of a group $w(G)$ to be $m(T)$ where T is the complete transfer system on a group.

Call a subgroup H of G "meet-irreducible" if it cannot be written as the intersection of two proper subgroups of G. Denote by $MI(G)$ the collection of meet irreducible subgroups.

For example, let G be a finite group, then

$$w(G) = |MI(G)/G|.$$

Another group invariant $c(G)$ the complexity could be the maximal value of $m(-)$. This is hard and mysterious, you can kinda relate it to something called a rainbow.

4.12 Michael Mandell : The tom Dieck splitting revisited

The goal is to describe a multiplicative analogue of the Tom-Dieck splitting for G-E_∞ algebras in equivariant homotopy theory. Joint work with Andrew Blumberg.
Let G be a finite group.

There are various equivariant stable homotopy categories, organised by a choice of universe, up to isotropy. There are two main theories.
Conisder only the trivial G-representation, then we call this the naive theory and we call the genuine theory, which is when we consider all finite dimensional representations.
The universe specifies which representations deloop. I am not quite sure what this means, but it was written on a slide, so I presume it's true.
There is the Steiner/little polydisk operad on universe. An equivalence of subcatories between group like $N - G$-spaces and connective G-spectra.
This is classical.

Nowadays you work with the algebras over a certain N_∞-operad.

Recall what an admissible set is.

There are equivalent characterizations for when G/K is admissible, which are internal to the equivariant stable homotopy theory we are studying, rather than needing to know the operad over which we are working. Such as compacity of $\Sigma^\infty_+ G/K$ or existence of a transfer (and others).

The Tom Dieck Splitting is a way to compute $(\Sigma^\infty_{\mathcal{N}} Z)^G$, where Z is some G-space, and $\mathcal{N}$ is the $\mathcal{N}_\infty$-operad with which we are working.
There is a comparison map with taking the fixed points before taking the suspension spectrum, assemblingt hese we get a map

$$\bigvee_{K \leq G} \Sigma^\infty((Z^K)_{hWK}) \to (\Sigma^\infty_{\mathcal{N}} Z)^G.$$

The Tom Dieck splitting is the statement that this map is a non-equivariant weak equivalence.

The $\mathcal{N}_\infty$-flavored suspension spectrum factors through the naive-G-equivariant flavored equivariant homotopy theory.
The maps used to "design" the Tom Dieck splitting map can be promotted to a naive G-equivariant weak equivalence. And this is a/the generalized Tom Dieck splitting.

The Tom Dieck splitting is a statement about change of $\mathcal{N}_\infty$ operad via pushforward.
We want to generalize it further to a multiplicative setting, and the maps used were (non-multiplicative) transfers, but the $\mathcal{N}_\infty$ technology also comes with multiplicative transfers. So we can do that.
Explicitly multiplicative transfers are constructed using the fact that $\mathcal{N}$ algebras admit a norm multiplication, you can take geometric fixed point. And then apply HHR. This gives a map from the

geometric fixed points for K to the geometric fixed points for G where $K \leq G$.
Then more details follow, but they flew over my head.

There was something about G-symmetric monoidal structures.

Then there was an "epilogue" about what happens when G is a compact lie group, rather than a finite group. It seems you still get Tom Dieck splitting.

4.13 Angelica Osorno : Equivariant operads, multiplicative structures, and transfer systems

Associative up to homotopy becomes A_n, especially at $n = \infty$ and commutative becomes E_n, especially at $n = \infty$. And notice $E_1 = A_\infty$.
And what should G-commutative? Let's work strict for a second
Let X be a G-space, for $H \leq G$, consider X^H, there are maps $X^H \to X^K$ if $K \leq H$ and there is a map $X^H \to H^{gHg^{-1}}$. If X is G-commutative (a monoid with a G-action or equivalently a monoid in G-spaces), then there also wrong way maps (transfer) $X^K \to X^H$, for $K \leq H$. And this is really thanks to the commutativity.

Now if we work up to homotopy, then to define G-commutative, say we consider E_∞ underlying the equivariant stuff. Then there are some choices in terms of which transfer maps we are requiring existence of, and THIS is the whole N_∞ business.
Let $K \leq H \leq G$ with $n = [H : K]$, and let $\Gamma \leq G \times S_n$, be a graph subgroup for some representation $H \to s_n$ of the H-set H/K.
For X an $\mathcal{O}$ algebra, we have a $G \times S_n$ map $\mathcal{O}(n) \to \mathrm{Map}_G(X^n, X)$. Taking Γ fixed point gives

$$\mathcal{O}(n)^\Gamma \to \mathrm{Map}_G(X^n \to X)^\Gamma \to \mathrm{Map}(X^K \to X^H)$$

And so exactly when $\mathcal{O}(n)^\Gamma$ is non empty we get transfer maps.

Theorem 4.13.1.

$$|Tr(C_{p^n})| = C_{n+1}$$

where C_{n+1} is a Catalan number.

Given an N_∞ operad you can define a transfer system, and this map gives an equivalence between the homotopy category of N_∞-operads for a fixed a group G and transfer systems for that group.
Given an N_∞-operad $\mathcal{O}$, then if you have an algebra in spaces, taking connected components gives an (incomplete) Mackey functor and if X is an $\mathcal{O}$ algebra in genuine G spectra, then taking connected components gives an (incomplete) tambara functor.

Then spoke a bit about compatible pairs, and how they relate to transfer systems and indexing systems.
Now given a compatible pair of transfer systems, which is a purely algebraic definition, one can ask if there is an operadis interpretation/realization? (natural thing to expect given where transfer systems come from).

Then definiiton of an operad pair, which is an action of an operad P on Q, but this is different from a module structure.

$$\lambda : P(k) \otimes Q(j_1) \otimes \cdots Q(j_k) \to Q(j_1 \cdots j_k).$$

Theorem 4.13.2. *(May) If P acts on Q, you can make sense of P-algebras in Q-algerbas (i.e. objects with a multiplicative P structure which distributes over an additive Q-structure).*

And a concrete question is if P and Q are both N_∞ operads thus the above compatibility relate in anyway to compatible pairs of the underlying transfer systems.
And it turns out that YES, by a result of the speaker and coauthors.

One problem, is that in practice finding two operads that act on each other is tricky. There is one famous example which is the linear isometries operad L acting on the Steiner operad K. Both L and K admit equivariant versions, and you can also make L act on K equivariantly. Which gives, for certains compatible paris of transfer systems, a way to realize the corresponding operads.

A cool general construction is to start with an operad in Set with free S_n action, then take $Map(G, -)$, then apply the fucntor $Set \to sSet$ which sends X the the simplicial set in level n is X^{n+1} which is a good way to construct operads in spaces.
This construction is sufficently functorial to preserve product pairs.
These ideas can be used to solve the realization problem for many pairs. And the strategy seems promising enough that it can be made to work for all compatible pairs.

4.14 Rekha Santhanam : Cofibrantly generated model structures for functor calculus

Consider a certain category of functors from $\mathcal{C} \to \mathcal{D}$, and we want to find n-excisive functor approximations.
And it would be nice to have model strcutures on $Fun(\mathcal{C}, \mathcal{D})$ that capture this notion of approximation, and this has been done.
Now, if $\mathcal{D}$ is cofibrantly generated, then there is a projective models structures on $Fun(\mathcal{C}, \mathcal{D})$, which is also cofibrantly generated.

Combining these two things we have a new wish. We want to find a model structure which is cofibrantly generated and such that weak equivalences are exactly $Q^{-1}(we)$ where Q is some P_n, i.e. a cofibrantly generated model structure which only sees up to n-excisiveness.

And in fact, by Bousfield localization, if Q is idempotent augmented and some other stuff, then you get all the above except cofibrantly generated, so somehow a fair amount of the work is done.

But getting cofibrant generation for a Bousfield localization is quite hard, and so we add a good handful of assumptions, which is not disappointing because these hold in general. The assumption seemingly serve to obtain some categorical tools needed for certain proofs.

The case in which we can do this is for "discrete" functor calculus, which is kinda like abelian functor calculus, but for functors of simplicial categories. One has to check a bunch of stuff, to apply the theory of localizations.

And another cool thing witht the model structure thus obtained is that the fibrant object are exactly the n-excisive functors.

The speaker gave a list of what categorical stuff we needed to prove the result, one important one was test morphisms, whose definition is too long for me to write up fast enough.

4.15 David De Mark : Compatibility of transfer systems and functoriality thereof

Very funny talk.

Recall Steiner and little disk operads.
Recall definition of a transfer system and relation to N_∞ operad.

Recall what it means for two transfer systems to be a compatible pair.

Given a subposet R of the poset of subgroups G a finite group, denote by $T(R)$ the smallest transfer system containing R.

A transfer system is disklike if it is generated by inclusions into the whole group, and saturated if it satisfies a condition which appeared in another talk.

Now the question is, given a disklike transfer system, what is the maximal pair with it?
The saturated hull $hull(T)$ of a transfer system, is the saturation of T.
A first observation is that the maximal compatible pair with a disklike transfer system must be saturated.
Another observation is that if a transfer system is already saturated, it is its own maximal pair.

A key technique in answering the question was to reframe the question using the complement of the maximal compatible counter part to a disklike transfer system.
This is quite a nice reframing, but quickly hard to check, but there is a hopeful conjecture, which has been checked in the abelian case and for order up to 64.

Another tool is a form of contravariant functoriality between transfer systems for different groups. Namely, the functor in question preserves disklike, saturated, compatible pairs if the second one is saturation and most interestingly it preserves maximal compatibility.

4.16 Kaif Hilman : Parametrised calculus and a spherical characterisation of parametrised stability

Equivariant calculus has been considered by Blumberg, Dotto-Moi and Dotto. Could it have applications to equivariant and hermition K-theory.
Nardin axiomatised something called Wirthmüller isomorphisms and mulitplicative norms via something called parametrised higher algebra. In particular defined a notion of parametrised stability.

Recall stability can be interpreted as saying that all the spheres act via equivalences. One can try to generalize this for example equivariantly by saying that $\mathcal{C}$ is G-stable if the G-representation spheres act via equivalences.
There is a slogan:
"Goodwillie calculus allows you to systematically tie spheres into semi-additivity. "

Definition 4.16.1. Let T be a category, we say it is orbital if Fin_T (finite coproduct completion of T) has pullbacks. We call T atomic if retracts are equivalences.
A T category $C \in Cat_T$ is an object of $Fun(T^{op}, Cat)$, there is a pointed vrsion.

Atomic orbitality axiomatises the double coset decompositions. A key example for most T is the functor that sends V to $\mathrm{Fun}(V, \mathcal{S})$ (I think, I might have missed something).

Let $J, C \in Cat_T$, in nice situations we have (fiberwise) left and right adjoints to $Fun(J, C) \leftarrow C$.

$V \in T$ can be viewed as a T category by some Yoneda lemma type thing.
A colimit over such an object of Cat_T is called a ...

For $f : U \to V$ a map in T and $\mathcal{C}$ a pointed T-category, there is a map between the adjoints of f^*, called the norm.
We call a T category C T-semi additive is ... and the norm is an equivalence, furthermore we call it

stable if it is fiberwise stable.

Question : What is a cube over T? This is painful because currying no longer allows for induction on the size of cubes based proof.
For any $f \in T$, there is a notion of f-cube in the T-parametrised setting, using left and right Kan extension mostly. In particular we can define Σ^f and Ω^f.
An interesting T over which to parametrise are locally short categoris, which is those such that for any V, the maximal chain of non-equivalence morphisms going to V is finite.

Theorem 4.16.2. *let T be locally short atomic orbital category and $\mathcal{C}$ a pointed T-presentable category then it is T stable if and only if it is spherically invertible.*

The proof idea is a generalization of a Goodwillie calculus based overkill proof that $\Sigma : \mathcal{C} \to \mathcal{C}$ being an equivalence implies semi-additivity (coproduct is a product). And then uses Robin Stoll's work on Goodwillie calculus for non cube shaped posets, known as (in this context at least) "excisable poset".
In this framework we can define cartesian and cocartesian relative to any exsiscable poset. And once you have that you can talk about excisvieness. We can also talk about a condition on categories which is like differentiable but just for a certain excisive-poset. And this is enough to recover an analog of the functor P_n, along with a nice formula for it.

4.17 Niall Taggart : Algebraic models for functor calculus

There is a theorem of Greenless and May, where rational G-equivariant spectra are equivalent to a certain product of chain complexes in a group ring.
And there is a dictionary between equivariant homotopy theory and calculus, so one might hope to repeat the above result for calculus.

In a sym monoidal presentable infinity category, an idempotent of the unit implies

$$\mathcal{C} \simeq Mod_{\mathcal{C}}(1) \simeq Mod_{\mathcal{C}}(e1) \oplus Mod_{\mathcal{C}}((Id - e)1).$$

There is a generalization of the burnside ring to a finite skeletal category, called Yoshida's abstract Burnside ring, which allows to do generalize some stuff from the equivariant picture to a more general framework.
The abstract burn side ring is denote $A_{\mathbb{Q}}(\mathcal{C})$ and there are details as the whether this is a ring, and is there even a ring structure? But I didn't have time to fully write this down.
One theorem is that a specific case where it works is if $\mathcal{C}$ has an epi-mono factorization.
This picture is rigid enough, that the fact that $A_{\mathbb{Q}}(C_2) \cong A_{\mathbb{Q}}(Epi^{\leq 2})$ implies isomorphism of certain corresonding categories (something equivariant for C_2 and something 2-excisive on the right).

In general, for $\mathcal{C} = Epi^{\leq n}$ is called the goodwillie Burnside ring, because functors out of it into a stable category are n-excisive functors from some category. We can compare the n-excisive to something equivariant, if we find some group whose burnside ring is the abstract burnside ring of $\mathcal{C}$. (at least I think that is the strategy, or at the very least evidence one might want to do that). I think the utility of the (abstract) Burnside ring is for determining idempotents.

Then there were some chain of equivalence of which a lot of them were "obvious" by quotting people, and one of the steps required a dream of a certain idempotent splitting to be nice enough. And it turns out, rationally things are nice enough that a lot of it can be solved by linear algebra over $\mathbb{Q}$ which works well.

The result in question is :

Theorem 4.17.1.

$$Exc_*^n(Sp^\omega, Sp)_\mathbb{Q} \simeq \prod_{1\leq k\leq n} Ch(\mathbb{Q}[\Sigma_k]).$$

Then there was an introduction to orthogonal calculus, which I listened to and didn't write down. One recent addition of the speaker and collaborators is an orthogonal analogue to the formula the classifies n-excisive functors as functors out of the category of epimorphisms and sets of size less than n.
(The slides went by real fast, it was hard to take notes)

4.18 Gjis Heuts : Koszul duality for En-operads and En-algebras

Let $Op(Sp)$ be the infinity category of operads in spectra. These are all non-unital. And everything is reduced.
Main object of interest in $\Sigma^\infty\mathbb{E}_n$.

Theorem 4.18.1. *The functor $Op(Sp)^{op} \to Cat_\infty/Sp$ which maps $\mathcal{O} \mapsto Alg_\mathcal{O}(Sp) \xrightarrow{U} Sp$ is a fully faithgul functor on $\mathbb{E}_n$ operads and their (de)suspensions.*

Remark 4.18.2. This result doesn't hold for formal reasons, in particular fails for many other operads. But it does mean that for $\mathbb{E}_n$ we can use their category of algebras to study them.

For $\mathcal{O} \in Op(Sp)$ there is a "suspension" $s\mathcal{O}$ such that

$$free_{s\mathcal{O}}(X) = \Sigma^{-1}free_\mathcal{O}(\Sigma X)$$

Note this is not the suspension in the category of operads.

Remark 4.18.3. 1. Note $s\mathcal{O}(n) \cong S^{\rho_n} \otimes \mathcal{O}(n)$
2. $s\mathcal{O}$ and $\mathcal{O}$ have the same categories of algebras.

Consider the functor $Bar : Alg^{aug}_{\mathbb{E}_n}(Sp) \to Alg^{aug}_{\mathcal{E}_{n-1}}(Sp)$. By the theorem (up to a small technicality), it corresponds to a map of operads $\beta : \mathbb{E}_n \to s\mathbb{E}_{n-1}$. (rhe s appears for technical reasons).
Notice this is kind of a wrong way map relative to what one might expect for the various $\mathbb{E}_n$.
Proof sketch of the existence of β, the bar construction has a right adjoint ω, you get a diagram for ω where the forgetful functors are compatible up to a shift, and so replacing one of the category of algebras by a shift, it becomes compatible with forgetful functor, and so the theorem at the beginning holds.

For $\mathbb{E}_n$ algebras there is Koszul duality, there are different things you can mean:
1. There is an adjunction $Bar^n \vdash Cobar^n$ between augmented algebras and coaugmented coalgebras.
2. There is a general notion of Koszul dual $B\mathcal{O}$ for operads $\mathcal{O}$ (seeing them as monoids) which you can specialize to $\mathbb{E}_n$ and see what happens. And in full generality you have an adjunction between the algebras over the operad and the coalgebras over the Koszul dual, the adjunction being $indec_\mathcal{O} \vdash prim_{B\mathcal{O}}$. The left adjoint $indec_\mathcal{O}$ can be thought of as a universal homology theory for $\mathcal{O}$-algebras by Basterra Mandel (I think).

Of course one expects these two perspectives to coincide. For that, one would need coAlgebras over $\mathbb{E}_n$ should be coalgebras over $B\mathbb{E}_n$, and so by the theorem at the top and the shifting construction, we expect $B\mathbb{E}_n \cong s^k\mathbb{E}_n$, and this was proven by Ching and Salvatore.
Furthermore, one can make the equivalence essentially unique with the property that it makes a certain (relatively natural) diagram commute. And this can be proven by highly non-geometric method, formal diagrams and stuff, whereas the other available proofs are very geometric. Though note that isn't the point, but rather that we get a sort of uniqueness.

Proof sketch : Consider graded spectra, and consider $\mathbb{E}_n$ algebras in graded spectra, call such a thing connected if the augmentation ideal is concentrated in strictly positive degrees.

Then one uses a theorem that shows that Bar-Cobar duality gives a equivalence of connected augmented (co)algebras in graded spectra. Now when one proves this, we want to commute a bar construction (realization) and a cobar construction (totalization), these tend not to commute, but in the graded case they do because everything becomes degree wise finite.
As a consequence of this we find an equivalence corresponding comonads, because the categories we showed are equivalent are comonadic over graded spectra. So the two we get are the same are $Sym_{B\mathbb{E}_n} \simeq Sym_{s^n\mathbb{E}_n^\vee}$. By a version of the theorem at the top, this gives an equivalence that shows that the $\mathbb{E}_n$ operad is Koszul self dual up to a shift.

Now the fact the equivalence we got is essentially unique, it is apparently easy to see that the map $\beta : \mathbb{E}_n \to s\mathbb{E}_{n-1}$ is Koszual dual to the inclusion. And the way one sees this is again by using the theorem at the top, and describing the maps we are interested in, in terms of what they do on the category of algebras.
What this allows is to study β (the wrong way map) via the relatively easy $\mathbb{E}_{n-1} \to \mathbb{E}_n$. For example take the limit of iterated β, a priori this is hard, but now we can instead take the Koszul dual of the colimit of the maps $\mathbb{E}_{n-1} \to \mathbb{E}_n$, which is $\mathbb{E}_\infty$ by definition, and so we get the spectral lie operad for the original problem.

This has applications to Poincaré-Birkhoff-Witt theorems, but for spectral lie algebras, and get $\mathbb{E}_n$-enveloping algebras.

Then there was a slide about a certain commutative square which is a "composition square", and "again", this statement can be verified by hand, but it is easier to use the theorem at the beginning and check something at the level of algebra categories.
You can use this composition square to build many more composition squares. In particular, taking an inverse limit, you can get composition squares involving the spectral lie operad. And you can go even further, with more inverse limits and you get

$$\mathbb{E}_1 \circ_{s\mathbb{L}} 1 \simeq \mathbb{E}_\infty.$$

Which is analogous of something clear in the classical case. And this is a satisfactory spectral restatement of PBW.

In fact there are plenty of corrolaries which allow you to understand/interpret PBW type results and envelopping algebras and that kinda stuff.

Proof sketch for the theorem at the beginning : It follows from the claim that the Schur functor is fully faithfull on $\mathbb{E}_n$ and their (de)suspensions.
For A, B symmetric sequences, denote by Sym_A, Sym_B the corresponding Schur functor, then one wants to show that given $f : Sym_A \to Sym_B$, it is built up from maps between "homogenous degree k parts". And this is the "meat and potatoes" of most of this talk from what I udnerstood. And what you need to prove this is to show that in the case of interest Sym_A and Sym_B have nilpotent euler class, and then use Goodwillie (dual) calculus.